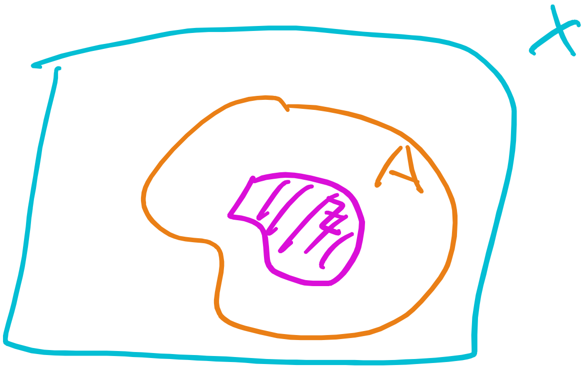
Alg Topo 4/9



Recall

① **Excision Thm**

Suppose

$$Z \subseteq A \subseteq X, \qquad cl(Z) \subseteq int(A)$$

Then the inclusion map

$$(X \setminus Z,\ A \setminus Z) \hookrightarrow (X, A)$$

induces iso

$$H_n(X \setminus Z,\ A \setminus Z) \xrightarrow{\cong} H_n(X, A) \quad \forall n$$

② If (X, A) is a good pair, then we have the following exact seq:

$$(*) \quad \boxed{\cdots \to \tilde{H}_n(A) \to \tilde{H}_n(X) \to \tilde{H}_n(X/A) \to \tilde{H}_{n-1}(A) \to \cdots}$$

③ We applied $(*)$ to $(X, A) = (D^n, \partial D^n)$

and proved

$$\tilde{H}_k(S^n) \cong \begin{cases} \mathbb{Z}, & k = n \\ 0, & k \neq n \end{cases}$$

Thm (Thm 2.26)

If nonempty open subsets $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ are homeomorphic, then $m = n$.

pf

Let x be an arbitrary point in U.

Applying Excision Thm to

$$(X, A, Z) = (\mathbb{R}^m, \underbrace{\mathbb{R}^m \setminus \{x\}}_{\text{open}}, \underbrace{\mathbb{R}^m \setminus U}_{\text{closed}})$$

we have

$$H_n(X, A) = H_n(\mathbb{R}^m, \mathbb{R}^m \setminus \{x\})$$

$$\cong H_n(X \setminus Z, A \setminus Z) = H_n(U, U \setminus \{x\})$$

By the exactness of

long exact seq for $(\mathbb{R}^m, \mathbb{R}^m \setminus \{x\})$

$$\cdots \to \tilde{H}_k(\mathbb{R}^m \setminus \{x\}) \to \boxed{\tilde{H}_k(\mathbb{R}^m)} \to H_k(\mathbb{R}^m, \mathbb{R}^m \setminus \{x\})$$

$$\to \tilde{H}_{k-1}(\mathbb{R}^m \setminus \{x\}) \to \boxed{\tilde{H}_{k-1}(\mathbb{R}^m)} \to \cdots$$

$\to H_{k-1}(\mathbb{R}^m \dots) \quad \boxed{\tilde{H}_{k-1}(\mathbb{R}^m)} = 0$

we have

$$\underline{H_k(\mathbb{R}^m, \mathbb{R}^m \setminus \{x\})} \cong \tilde{H}_{k-1}(\boxed{\mathbb{R}^m \setminus \{x\}})$$

homotopy equivalent to S^{m-1}

$$\cong \tilde{H}_{k-1}(S^{m-1}) \cong \underline{\begin{cases} \mathbb{Z}, & k-1 = m-1 \Leftrightarrow k = m \\ 0, & k \neq m \end{cases}}$$

So

$$H_k(U, U \setminus \{x\}) \cong \begin{cases} \mathbb{Z}, & k = m \\ 0, & k \neq m \end{cases}$$

Similarly, for $y \in V \underset{\text{open}}{\subseteq} \mathbb{R}^n$,

$$H_k(V, V \setminus \{y\}) \cong \begin{cases} \mathbb{Z}, & k = n \\ 0, & k \neq n \end{cases}$$

Assume

$$\phi : U \to V$$

is a homeomorphism. Then

$$\phi|_{U \setminus \{x\}} : U \setminus \{x\} \xrightarrow{\text{homeomorphism}} V \setminus \{\phi(x)\}$$

$\phi|_{U\setminus\{x\}}$

and

$$\phi : (U, U\setminus\{x\}) \longrightarrow (V, V\setminus\{\phi(x)\})$$

are homeomorphisms.

$\Rightarrow$
$$H_k(U, U\setminus\{x\}) \cong H_k(V, V\setminus\{\phi(x)\}) \quad \forall k$$

$\Rightarrow \quad n = m \qquad\qquad \#$

Naturality and axiomatic homology

Relative homology is natural in the following sense:

Prop

Let $A \subset X$ and $B \subset Y$. A map
$$f : (X, A) \longrightarrow (Y, B)$$
of pairs induces the following commutative diagram:

$$\cdots \longrightarrow H_n(A) \longrightarrow H_n(X) \longrightarrow H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \rightarrow \cdots$$

$$\begin{tikzcd}
{} \arrow[d, "f'_*"'] & {} \arrow[d, "f_*"'] & {} \arrow[d, "f_*"'] & {} \arrow[d, "f'_*"'] \\
\cdots \arrow[r] & H_n(B) \arrow[r] & H_n(Y) \arrow[r] & H_n(Y,B) \arrow[r, "\partial"'] & H_{n-1}(B) \arrow[r] & \cdots
\end{tikzcd}$$

To prove this proposition, we need

Lemma

Assume we have a "morphism of short exact sequences of chain complexes"

$$\begin{tikzcd}
0 \arrow[r] & (A,\partial^A) \arrow[r, "i"] \arrow[d, "\alpha"] \arrow[dr, phantom, "\circlearrowright"] & (B,\partial^B) \arrow[r, "j"] \arrow[d, "\beta"] \arrow[dr, phantom, "\circlearrowright"] & (C,\partial^C) \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & (A',\partial^{A'}) \arrow[r] & (B',\partial^{B'}) \arrow[r] & (C',\partial^{C'}) \arrow[r] & 0
\end{tikzcd}$$

chain maps, commute with i, j

Then the induced diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(A) \arrow[r] \arrow[d, "\alpha_*"] & H_n(B) \arrow[r] \arrow[d, "\beta_*"] & H_n(C) \arrow[r] \arrow[d, "\gamma_*"] & H_{n-1}(A) \arrow[r] \arrow[d, "\alpha_*"] & \cdots \\
\cdots \arrow[r] & H_n(A') \arrow[r] & H_n(B') \arrow[r] & H_n(C') \arrow[r] & H_{n-1}(A') \arrow[r] & \cdots
\end{tikzcd}$$

Commute.

proof of Prop

A map $f:(X,A)\to(Y,B)$ of pairs

induces a morphism of short exact seq of

$$
\begin{tikzcd}
0 \arrow[r] & C_\bullet(A) \arrow[r] \arrow[d, "f_\#"'] & C_\bullet(X) \arrow[r] \arrow[d, "f_\#"'] & C_\bullet(X,A) = C_\bullet(X)/C_\bullet(A) \arrow[r] \arrow[d, "f_\#"'] & 0 \\
0 \arrow[r] & C_\bullet(B) \arrow[r] & C_\bullet(Y) \arrow[r] & C_\bullet(Y,B) \arrow[r] & 0
\end{tikzcd}
$$

So Lemma $\Rightarrow$ Prop. #

<u>Cor</u>

For a map $f:(X,A) \to (Y,B)$ of good pairs, the diagram

$$
\begin{tikzcd}
\cdots \arrow[r] & \tilde{H}_n(A) \arrow[r] \arrow[d, "f_*"] & \tilde{H}_n(X) \arrow[r] \arrow[d, "f_*"] & \tilde{H}_n(X/A) \arrow[r] \arrow[d, "f_*"] & \tilde{H}_{n-1}(A) \arrow[r] & \cdots \\
\cdots \arrow[r] & \tilde{H}_n(B) \arrow[r] & \tilde{H}_n(Y) \arrow[r] & \tilde{H}_n(Y/B) \arrow[r] & \tilde{H}_{n-1}(B) \arrow[r] & \cdots
\end{tikzcd}
$$

commutes.

The proposition is the last piece of homology theory in the sense of Eilenberg-Steenrod axioms:

Def.

A category $\mathcal{C}$ consists of

- $Ob(\mathcal{C})$ — objects of $\mathcal{C}$,
- $Mor(X,Y)$ — morphisms from a source object X to a target object Y,
- $\circ$ — composition of morphisms

$$\circ : Mor(Y,Z) \times Mor(X,Y) \to Mor(X,Z)$$

such that the following properties hold:

(i) Associativity:

$\forall\ f \in Mor(X,Y),\ g \in Mor(Y,Z),\ h \in Mor(Z,U),$

$$h \circ (g \circ f) = (h \circ g) \circ f$$

(ii) Identity:

$\forall\ X \in Ob(\mathcal{C}),\ \exists\ id_X \in Mor(X,X)$, called the identity morphism for X, s.t.

$\forall f \in Mor(X, Y)$,

$$id_Y \circ f = f = f \circ id_X .$$

We also write $f: X \to Y$ if $f \in Mor(X, Y)$

Example

① Category of abelian groups.

objects = abelian groups

mor. = group homomorphisms

composition = usual composition

② Category of topological spaces.

objects = topological spaces

mor. = continuous maps

composition = the usual one.

③ Category of pairs of spaces.

objects = (X, A). X: space, $A \subset X$.

mor = maps of pairs — $f: (X, A) \to (Y, B)$

$f: X \to Y$, $f(A) \subseteq B$

composition = usual one.

Def

A (covariant) functor F from a category $\mathcal{C}$ to another category $\mathcal{D}$, written $F: \mathcal{C} \to \mathcal{D}$, consists of:

resp. contravariant functor G

- $\forall X \in Ob(\mathcal{C})$, we have $F(X) \in Ob(\mathcal{D})$
- $\forall f \in Mor_{\mathcal{C}}(X, Y)$, we have $G(f) \in Mor_{\mathcal{D}}(G(Y), G(X))$

$$F(f) \in Mor_{\mathcal{D}}(F(X), F(Y))$$

s.t. the following properties hold:

(i) $\forall X \in Ob(\mathcal{C})$, $F(id_X) = id_{F(X)}$

(ii) $\forall$ morphisms $f: X \to Y$, $g: Y \to Z$ in $\mathcal{C}$, we have $G(g \circ f) = G(f) \circ G(g)$

$$F(g \circ f) = F(g) \circ F(f)$$

$F(X) \to F(Z)$ mor in $\mathcal{D}$

Example

① Homology groups:

For each n,

H_n: cat. of spaces $\longrightarrow$ cat. of abelian gp spaces ... of

is a functor. $(H_n(f) = f_*)$

Recall:

$$(g \circ f)_* = g_* \circ f_*$$

$$(id_X)_* = id_{H_n(X)}$$

② Reduced homology groups:

$$\tilde{H}_n : \text{cat. of spaces} \longrightarrow \text{cat. of ab. gp.}$$

③ Relative homology groups:

$$H_n : \text{cat. of pairs of spaces} \longrightarrow \text{cat. of ab. gp.}$$

$$(X, A) \longmapsto H_n(X, A)$$

Remark

A mor. $f: X \to Y$ in a category $\mathcal{C}$ is called an isomorphism if $\exists$ mor

$g: Y \to X$ s.t.

$$f \circ g = id_Y$$

$$g \circ f = id_X$$

inverse of f

Claim:

If $F: \mathcal{C} \to \mathcal{D}$ is a functor and

$\phi: X \to Y$ is an isomorphism in $\mathcal{C}$,

then $F(\phi): F(X) \to F(Y)$ is an isomorphism in $\mathcal{D}$.

pf

Assume $\phi: X \to Y$ is an isomorphism in $\mathcal{C}$

i.e. $\exists \overset{\text{mor}}{\psi}: Y \to X$ in $\mathcal{C}$ s.t.

$$\phi \circ \psi = id_Y, \qquad \psi \circ \phi = id_X$$

Then

$$\begin{aligned} F(\phi \circ \psi) &= F(id_Y) = id_{F(Y)} \\ \| \quad & \\ F(\phi) \circ F(\psi) & \end{aligned}$$

and

$$F(\psi) \circ F(\phi) = F(\psi \circ \phi) = F(id_X) = id_{F(X)}$$

So $F(\phi)$ is an iso in $\mathcal{D}$

(with inverse $F(\psi)$) #

Def'

Let F and G be functors from $\mathcal{C}$ to $\mathcal{D}$.

A natural transformation η from F to G associates each $X \in Ob(\mathcal{C})$ a mor.

$$\eta_X : F(X) \longrightarrow G(X) \quad \text{— mor in } \mathcal{D}$$

s.t. $\eta_Y \circ F(f) = G(f) \circ \eta_X \quad \forall f \in Mor(X, Y)$

i.e. the following diagram commutes:

$$\begin{tikzcd}
F(X) \arrow[r, "F(f)"] \arrow[d, "\eta_X"'] & F(Y) \arrow[d, "\eta_Y"] \\
G(X) \arrow[r, "G(f)"'] & G(Y)
\end{tikzcd}$$

Example

The connecting homomorphism

$$\partial : H_n(X, A) \longrightarrow H_{n-1}(A)$$

is a natural transformation from the functor of F n-th relative homology group to the functor

$$G : (X, A) \longmapsto H_{n-1}(A)$$

This means:

$$\begin{tikzcd} H_n(X,A) \arrow[r, "f_*"] \arrow[d, "\partial_{(X,A)}"'] & H_n(Y,B) \arrow[d, "\partial_{(Y,B)}"] \\ H_{n-1}(A) \arrow[r, "f_*"'] & H_{n-1}(B) \end{tikzcd}$$

— It is a consequence of our Prop

Def (Eilenberg–Steenrod axioms)

A homology theory is a sequence of functors

$$H_n : \text{the cat. of pairs of spaces} \longrightarrow \text{the cat. of abelian groups}$$

together with a natural transformation

$$\partial : H_n(X, A) \longrightarrow H_{n-1}(A) \overset{\text{def.}}{=\!=} H_{n-1}(A, \emptyset)$$

which satisfies the following axioms:

(i) Homotopy:

Homotopic (through maps of pairs) maps

homotopic (through maps of pairs)
induce the same map in homology

(ii) <u>Excision</u>:

If $Z \subset A \subset X$ s.t. $\mathrm{cl}(Z) \subset \mathrm{int}(A)$,
then the inclusion map

$$(X \setminus Z, A \setminus Z) \hookrightarrow (X, A)$$

induces iso in homology

(iii) <u>Dimension</u>:

If $*$ is the one-point space, then

$$H_n(*) = 0 \quad \forall\, n \neq 0.$$

(iv) <u>Additivity</u>:

If $X = \coprod_\alpha X_\alpha$, then

$$H_n(X) \cong \bigoplus_\alpha H_n(X_\alpha)$$

(v) Each pair (X, A) induces a long exact seq. in homology via the inclusion maps:

$$i : A \hookrightarrow X$$

$$j : X = (X, \emptyset) \hookrightarrow (X, A)$$

and the natural transformation ∂

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Thm

Singular homology is a homology theory.

Remark

① Many facts about homology, such as homology of S^n, can be derived directly from the axioms.

For some nice spaces, such as CW complexes, one has a uniqueness thm for homology theory.

② A "homology-like" theory satisfying all the axioms except the dimension axiom is called an extraordinary homology theory (dually, extraordinary homology theory (dually, extraordinary cohomology theory).

Important examples of extraordinary cohomology theory were found in 1950s such as

topological K-theory

and

cobordism theory

and come with homology theories dual to them.

$$
\begin{tikzcd}
 & & 0 \arrow[dd] & & 0 \arrow[dd] & & 0 \arrow[dd] & \\
 & 0 \arrow[dd] & & 0 \arrow[dd] & & 0 \arrow[dd] & & \\
 & \cdots \arrow[r] & A'_{n+1} \arrow[rr, "\partial"] \arrow[dd, dashed, "i'"] & & A'_n \arrow[rr, "\partial"] \arrow[dd, dashed, "i'"] & & A'_{n-1} \arrow[r] \arrow[dd, dashed, "i'"] & \cdots \\
\cdots \arrow[r] & A_{n+1} \arrow[ur, "\alpha"] \arrow[rr, "\partial"] \arrow[dd, "i"] & & A_n \arrow[ur, "\alpha"] \arrow[rr, "\partial"] \arrow[dd, "i"] & & A_{n-1} \arrow[ur, "\alpha"] \arrow[r] \arrow[dd, "i"] & \cdots & \\
 & \cdots \arrow[r] & B'_{n+1} \arrow[rr, "\partial"] \arrow[dd, dashed, "j'"] & & B'_n \arrow[rr, "\partial"] \arrow[dd, dashed, "j'"] & & B'_{n-1} \arrow[r] \arrow[dd, dashed, "j'"] & \cdots \\
\cdots \arrow[r] & B_{n+1} \arrow[ur, "\beta"] \arrow[rr, "\partial"] \arrow[dd, "j"] & & B_n \arrow[ur, "\beta"] \arrow[rr, "\partial"] \arrow[dd, "j"] & & B_{n-1} \arrow[ur, "\beta"] \arrow[r] \arrow[dd, "j"] & \cdots & \\
 & \cdots \arrow[r] & C'_{n+1} \arrow[rr, "\partial"] \arrow[dd, dashed] & & C'_n \arrow[rr, "\partial"] \arrow[dd, dashed] & & C'_{n-1} \arrow[r] \arrow[dd, dashed] & \cdots \\
\cdots \arrow[r] & C_{n+1} \arrow[ur, "\gamma"] \arrow[rr, "\partial"] \arrow[dd] & & C_n \arrow[ur, "\gamma"] \arrow[rr, "\partial"] \arrow[dd] & & C_{n-1} \arrow[ur, "\gamma"] \arrow[r] \arrow[dd] & \cdots & \\
 & & 0 & & 0 & & 0 & \\
 & 0 & & 0 & & 0 & &
\end{tikzcd}
$$