Alg Topo 4/2

Recall

$$\tilde{H}_k(S^n) \cong \begin{cases} \mathbb{Z} & k = n \\ 0 & k \neq n \end{cases}$$

Cor

$$\mathbb{R}^m \cong \mathbb{R}^n \iff m = n$$

pf

Assume $\Phi : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is a homeomorphism.

$$\Rightarrow \quad \Phi\big|_{\mathbb{R}^m \setminus \{\vec{0}\}} : \mathbb{R}^m \setminus \{\vec{0}\} \longrightarrow \mathbb{R}^n \setminus \{\Phi(\vec{0})\}$$

is also a homeomorphism

$$\Rightarrow \quad \tilde{H}^k(\mathbb{R}^m \setminus \{\vec{0}\}) \cong \tilde{H}^k(\mathbb{R}^n \setminus \{\Phi(\vec{0})\}) \quad \forall k$$

__Note__

$$\begin{tikzcd} S^m = \{\vec{x} \in \mathbb{R}^m \mid \|\vec{x}\| = 1\} \arrow[r, hook, "i"] & \mathbb{R}^m \setminus \{\vec{0}\} \arrow[l, bend left, "r"] \end{tikzcd}$$

$$\frac{\vec{x}}{\|\vec{x}\|} \overset{r}{\longmapsfrom} \vec{x}$$

$$r \circ i = id_{S^m}$$

$$i \circ r \sim_F id_{\mathbb{R}^m \setminus \{\vec{0}\}}, \quad \text{where}$$

$$F : (\mathbb{R}^m \setminus \{\vec{0}\}) \times I \longrightarrow \mathbb{R}^m \setminus \{\vec{0}\}$$

$$F(\vec{x}, t) = \frac{\vec{x}}{t + (1-t)\|\vec{x}\|}$$

$$F(\vec{x}, 0) = \frac{\vec{x}}{\|\vec{x}\|} = i \circ r(\vec{x})$$

$$F(\vec{x}, 1) = \vec{x} = id(\vec{x})$$

$\Rightarrow$ S^m and $\mathbb{R}^m \setminus \{\vec{0}\}$ are homotopy equivalent

$\Rightarrow$ $\tilde{H}_k(S^m) \cong \tilde{H}_k(\mathbb{R}^m \setminus \{\vec{0}\}) \quad \forall k.$

Therefore,

$$\begin{array}{ccc} \tilde{H}_k(\mathbb{R}^m \setminus \{\vec{0}\}) & \cong & \tilde{H}_k(\mathbb{R}^n \setminus \{\phi(\vec{0})\}) \\ \wr\!\shortparallel & & \wr\!\shortparallel \\ \tilde{H}_k(S^m) & & \tilde{H}_k(S^n) \end{array}$$

$H_k(S^m) \qquad\qquad\qquad H_k(S^n)$

$\Rightarrow \quad \tilde{H}_n(S^m) \cong \tilde{H}_n(S^n) \cong \mathbb{Z}$

$\Rightarrow \quad n = m \qquad\qquad \#$

<u>Recall</u>

Assume X is a topological space $A \subseteq X$.

$\Rightarrow \quad C_n(A) \subseteq C_n(X)$

$$\Rightarrow \quad \begin{array}{ccc} C_n(X,A) & = & C_n(X)/C_n(A) \\ \downarrow \partial_n^{X,A} & & \downarrow \partial_n^X \\ C_{n-1}(X,A) & = & C_{n-1}(X)/C_{n-1}(A) \end{array}$$

$$\boxed{H_n(X,A) = \ker(\partial_n^{X,A}) / \operatorname{im}(\partial_{n+1}^{X,A})}$$

$C_\bullet(X,A)$

$$0 \to C_\bullet(A) \longrightarrow C_\bullet(X) \longrightarrow \overset{\shortparallel}{C_\bullet(X)/C_\bullet(A)} \to 0$$

is a short exact seq. of chain cxes

$\Rightarrow$ we have

<u>Thm</u>

Following are exact:

$$\cdots \to H_n(A) \to H_n(X) \to H_n(X,A) \to H_{n-1}(A) \to \cdots,$$

and

$$\cdots \to \tilde{H}_n(A) \to \tilde{H}_n(X) \to \tilde{H}_n(X,A) \to \tilde{H}_{n-1}(A) \to \cdots$$

where

$$\tilde{H}_n(X,A) := \begin{cases} H_n(X,A) & \text{if } A \neq \emptyset \\ \tilde{H}_n(X) & \text{if } A = \emptyset \end{cases}$$

$$\cdots \to \overset{0}{\cancel{\tilde{H}_0(\emptyset)}} \to \tilde{H}_0(X) \xrightarrow{\cong} \tilde{H}_0(X,\emptyset) \to 0$$

$$\tilde{H}_0(X,\emptyset) = \tilde{H}_0(X)$$

$$\tilde{C}_\bullet(X,A) = \tilde{C}_\bullet(X)/\tilde{C}_\bullet(A)$$

$\tilde{C}_\bullet(X)$:
$$\cdots \to C_1(X) \to C_0(X) \xrightarrow{\varepsilon} \mathbb{Z}$$

Example

For $(X, A) = (D^n, \partial D^n)$,

$$\cdots \to \underbrace{\boxed{\tilde{H}_k(D^n)}}_{=0} \to \tilde{H}_k(D^n, \partial D^n) \xrightarrow{\cong} \tilde{H}_{k-1}(\partial D^n) \to \underbrace{\boxed{\tilde{H}_{k-1}(D^n)}}_{=0} \to \cdots$$

$$\Rightarrow \quad \tilde{H}_k(D^n, \partial D^n) \cong \tilde{H}_{k-1}(\partial D^n) \cong \tilde{H}_{k-1}(S^{n-1}) \quad \forall k$$

#

Example

Let $x_0 \in X$, $\quad (X, A) = (X, \{x_0\})$

$$\Rightarrow \quad \cdots \to \underbrace{\boxed{\tilde{H}_n(\{x_0\})}}_{=0} \to \tilde{H}_n(X) \xrightarrow{\cong} \tilde{H}_n(X, \{x_0\}) \to \underbrace{\boxed{\tilde{H}_{n-1}(\{x_0\})}}_{=0} \to \cdots$$

So
$$H_n(X, \{x_0\}) = \tilde{H}_n(X, \{x_0\}) \cong \tilde{H}_n(X)$$

#

Def

We write

$f : (X, A) \longrightarrow (Y, B)$

if $f : X \to Y$ is a continuous map

s.t. $f(A) \subseteq B$.

$\Rightarrow$ $f_\# : C_n(X) \longrightarrow C_n(Y)$

$f_\#|_{C_n(A)} : C_n(A) \longrightarrow C_n(B)$

$\Rightarrow$ $f_\# : \begin{matrix} C_n(X)/C_n(A) \\ \| \\ C_n(X, A) \end{matrix} \longrightarrow \begin{matrix} C_n(Y)/C_n(B) \\ \| \\ C_n(Y, B) \end{matrix}$

is a chain map

$\Rightarrow$

$f_* : H_n(X, A) \longrightarrow H_n(Y, B)$

<u>Def</u>

Two maps

two maps

$$f, g : (X,A) \longrightarrow (Y,B)$$

are homotopic through maps $(X,A) \longrightarrow (Y,B)$ if $\exists$ continuous map

$$H : X \times I \longrightarrow Y$$

s.t.

$$H(A \times I) \subseteq B$$

$$H(x,0) = f(x), \quad H(x,1) = g(x).$$

<u>Prop</u>

If two maps

$$f, g : (X,A) \longrightarrow (Y,B)$$

are homotopic through maps $(X,A) \to (Y,B)$, then

$$f_* = g_* : H_n(X,A) \longrightarrow H_n(Y,B) \quad \forall n.$$

sketch of pf

Sketch of pf

Consider the chain homotopy P constructed in the proof of the homotopy invariance thm.

The assumption $\Rightarrow$ $P(C_\cdot(A)) \subseteq C_{\cdot+1}(B)$

$\rightsquigarrow$ chain homotopy between

$$f_\#, g_\#: C_\cdot(X,A) \to C_\cdot(Y,B).$$ $\#$

Remark

If $B \subseteq A \subseteq X$, then

$$0 \to C_\cdot(A,B) \to C_\cdot(X,B) \to C_\cdot(X,A) \to 0$$

is a short exact seq. of cx

$\Rightarrow$

$$\cdots \to H_n(A,B) \to H_n(X,B) \to H_n(X,A) \to H_{n-1}(A,B) \to \cdots$$

is exact.

__Thm__ (Excision Thm. Thm 2.20)

Suppose $Z \subseteq A \subseteq X$ s.t.

$$\bar{Z} = \text{closure}(Z) \subseteq \mathring{A} = \text{interior}(A)$$

Then the inclusion map

$$(X \setminus Z, A \setminus Z) \hookrightarrow (X, A)$$

induces iso

$$H_n(X \setminus Z, A \setminus Z) \xrightarrow{\cong} H_n(X, A) \quad \forall n.$$

Equivalently, if A and B are subspaces of X with the property

$$X = \text{int}(A) \cup \text{int}(B),$$

then the inclusion map

$$(B, A\cap B) \hookrightarrow (X, A)$$

induces iso

$$H_n(B, A\cap B) \xrightarrow{\cong} H_n(X,A) \quad \forall n$$

<u>key idea for proving Excision Thm</u>:

"barycentric subdivision"

Let

$$\mathcal{U} = \{U_\lambda\}$$

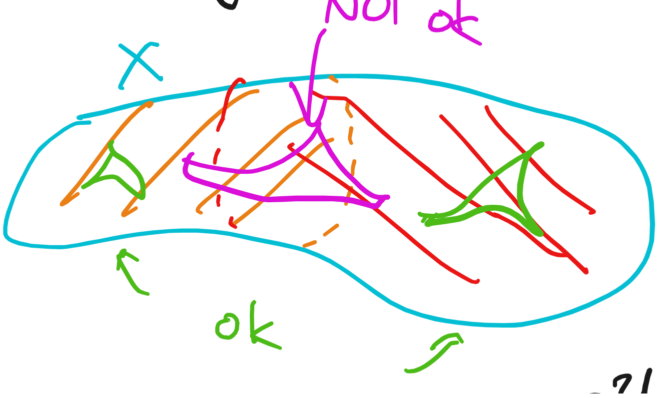
be a collection of subsets of X

s.t.

$$\bigcup_\lambda \mathrm{int}(U_\lambda) = X$$

Define

$$C_n^{\mathcal{U}}(X) = \left\{ \sum m_\sigma \cdot \sigma \in C_n(X) \;\middle|\; \begin{array}{l} \sigma : \Delta^n \to X \\ \sigma(\Delta^n) \subseteq U_\lambda \\ \text{for some } U_\lambda \in \mathcal{U} \end{array} \right\}$$



$$\begin{array}{c} \cap\!\mid \\ C_n(X) \\ \downarrow \partial_n^X \end{array}$$

$$C^{\mathcal{U}}_{n-1}(X) \subseteq C_{n-1}(X)$$

Since

$$\partial^X_n(C^{\mathcal{U}}_n(X)) \subseteq C^{\mathcal{U}}_{n-1}(X)$$

we have the chain complex

$$(C^{\mathcal{U}}_\bullet(X), \partial^{\mathcal{U}}).$$

Its n-th homology is denoted by $H^{\mathcal{U}}_n(X)$.

<u>Prop</u>

The inclusion map

$$i: C^{\mathcal{U}}_\bullet(X) \longrightarrow C_\bullet(X)$$

is a chain homotopy equivalence, i.e.

$\exists$ another chain map

$$j: C_\bullet(X) \longrightarrow C^{\mathcal{U}}_\bullet(X)$$

s.t. $i \circ j$ is chain homotopic to $id_{C_\bullet(X)}$

$j \circ i$ " " $id_{C^{\mathcal{U}}_\bullet(X)}$

In particular,

$$H_n^{\mathcal{U}}(X) \cong H_n(X) \qquad \forall n$$

<u>sketch of pf</u> (page 119 – 124)

<u>Step1</u>

$$\Delta^N = \{(t_0, \cdots, t_N) \in \mathbb{R}^{N+1} \mid t_0 + \cdots + t_N = 1,\ t_i \geq 0\}$$

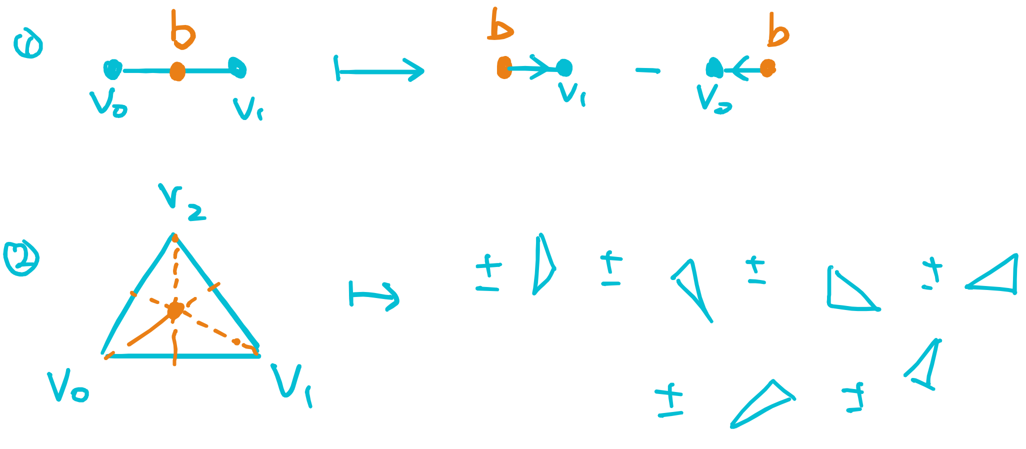
is a convex subset of $\mathbb{R}^{N+1}$

Consider " $C_n^{\text{linear}}(\Delta^N)$ " and define

"barycentric subdivision" homomorphism

$$S : C_n^{\text{lin}}(\Delta^N) \to C_n^{\text{lin}}(\Delta^N)$$

picture of S :



<u>Step2</u>

Define a homotopy operator

$$T : C_n^{\text{lin}}(\Delta^N) \to C_{n+1}^{\text{lin}}(\Delta^N)$$

s.t.

$$\partial T + T\partial = id - S$$

$\Rightarrow \quad S_* = id$ on homology

<u>Step2</u>

Define analogous operators for singular chains.

For $\sigma : \Delta^n \to X$, consider

$$\Delta^n \xrightarrow{id} \Delta^n \xrightarrow{\sigma} X$$

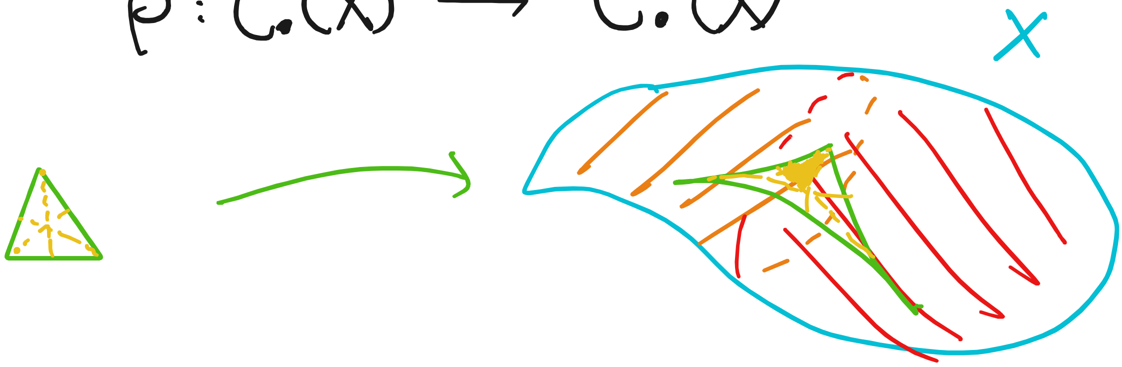
Apply S, T to

$\rightsquigarrow$ S, T on $C_n(X)$.

<u>Step4</u>

Apply S, T many times:

$$\rho : C_\bullet(X) \longrightarrow C_\bullet^{\mathcal{U}}(X)$$



and a chain homotopy

$i : C_\bullet^{\mathcal{U}}(X) \hookrightarrow C_\bullet(X)$

$$D : C_\bullet(X) \longrightarrow C_\bullet(X)$$

s.t. $\partial D + D\partial = id - ip.$

$p \circ i = id$

$\Rightarrow$ Prop. #

pf of Exicision Thm

Let $\mathcal{U} = \{A, B\}$

$\exists$ $\rho : C_\bullet(X) \to C_\bullet^{\mathcal{U}}(X)$

D : homotopy operator.

Note:

$\rho(C_\bullet(A)) \subseteq C_\bullet(A)$,

$D(C_\bullet(A)) \subseteq C_{\bullet+1}(A)$

$\Rightarrow \exists\ \rho, D$ on $C_\bullet^{\mathcal{U}}(X) / C_\bullet(A)$ and $C_\bullet(X) / C_\bullet(A)$

Here: ??

$\Rightarrow$ $\boxed{C_\bullet^{\mathcal{U}}(X) / C_\bullet(A)} \hookrightarrow C_\bullet(X) / C_\bullet(A)$

is a chain homotopy equivalence.

Since (by definition)

$$C_\bullet(B) \big/ C_\bullet(A\cap B) \quad \text{and} \quad C_\bullet^{\mathcal{U}}(X) \big/ C_\bullet(A)$$

are both the free abelian group generated by $\sigma : \Delta^n \to B$ s.t. $\sigma(\Delta^n) \not\subseteq A$, their homology groups are naturally identified.

So

$$H_n(B, A\cap B) \cong H_n\left(C_\bullet^{\mathcal{U}}(X) \big/ C_\bullet(A) \right)$$

$$\cong H_n(X, A) \qquad \#$$

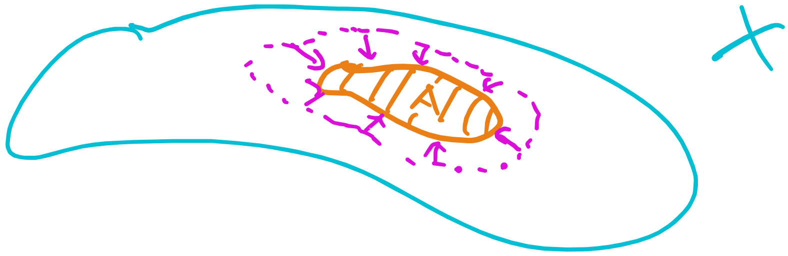
<u>Recall</u>

We need: a space X together with nonempty closed subset A that is a deformation retract of some neighborhood of A in X.

Such a pair (X, A) is called

Such a pair (X,A) is called

a good pair. in textbook.



Prop

For any good pair (X,A), the quotient map

$$q : (X,A) \longrightarrow (X/A\, ,\, A/A)$$

induces iso

$$q_* : H_n(X,A) \xrightarrow{\cong} H_n(X/A\, ,\, A/A) \cong \tilde{H}_n(X/A) \qquad \forall n$$

pf

Let V be a neighborhood of A in X that deformation retracts onto A.

We have the commutative diagram

$$H_n(X,A) \xrightarrow{\cong} H_n(X,V) \xleftarrow{\cong} H_n(X \setminus A,\, V \setminus A)$$

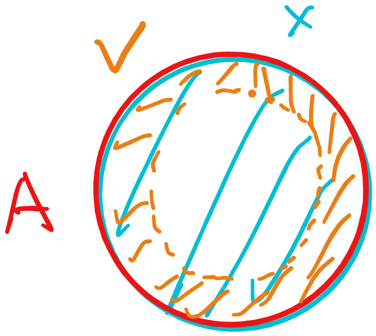
Conclusion:

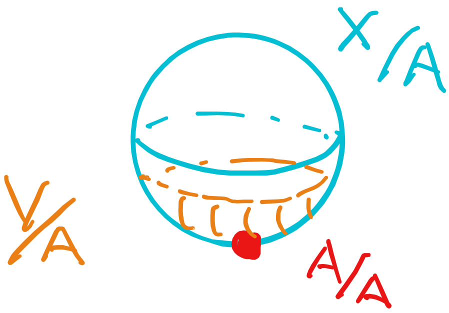
$$\begin{array}{ccccc} \downarrow\cong & & \downarrow & & \downarrow\cong \\ H_n(X/A, A/A) & \xrightarrow{\cong} & H_n(X/A, V/A) & \xleftarrow{\cong} & H_n(X/A \setminus A/A, V/A \setminus A/A) \end{array}$$

(same reason)

Excision Thm

induced by a homeomorphism





① Consider the long exact seq. for (X, V, A)

$$\cdots \to \boxed{H_n(V,A)} \to H_n(X,A) \xrightarrow[\cong]{①} H_n(X,V) \to \boxed{H_{n-1}(V,A)} \to \cdots$$

$\boxed{H_n(V,A)} = 0$, $\boxed{H_{n-1}(V,A)} = 0$

Since V deformation retracts onto A,

$$H_n(V,A) \cong H_n(A,A) = 0 \quad \forall n.$$

Recall:

$i: A \hookrightarrow V$, $\quad r: V \to A$ $\qquad H(A \times I) \subseteq A$

homotopy through $(V,A) \to (V,A)$

$r \circ i = id_A$

$i \circ r \sim_H id_V$, and $H(a,t) = a \quad \forall t \in I \ \forall a \in A$

$$\begin{tikzcd} H_n(V,A) \arrow[r, "r_*", "\cong"'] \arrow[rr, bend right, "id"'] & H_n(A,A) \arrow[r, "i_*"] & H_n(V,A) \end{tikzcd}$$

#

Recall

We used the exact seq.

$$(*) \quad \cdots \to \tilde{H}_n(A) \to \tilde{H}_n(X) \to \boxed{\tilde{H}_n(X/A)} \to \tilde{H}_{n-1}(A) \to \cdots$$

for good pair (X, A).

Note:

$(*)$ is exact if $A = \phi$

Assume $A \neq \phi$.

Recall we have the exact seq.

$$\cdots \to \tilde{H}_n(A) \to \tilde{H}_n(X) \to \boxed{\tilde{H}_n(X, A)} \to \tilde{H}_{n-1}(A) \to \cdots$$

$\| \; A \neq \phi$

$H_n(X, A)$

$\cong$

$\tilde{H}_n(X/A)$

$\Rightarrow$ $(*)$ is exact. #