# Alg Topo 3/26

<u>Recall</u>

If two maps $f, g : X \to Y$ are homotopic, then they induce the same homomorphism

$$f_* = g_* : H_n(X) \longrightarrow H_n(Y) \quad \forall n$$

<u>Cor</u>

If $f : X \to Y$ is a homotopy equivalence, then

$$f_* : H_n(X) \longrightarrow H_n(Y)$$

is an isomorphism for each $n$.

<u>pf</u>

$f^{:X\to Y}$ is a homotopy equi.

... a homotopy eq

$\Rightarrow \exists\ g : Y \to X$ s.t.

(i) $f \circ g \sim id_Y$

(ii) $g \circ f \sim id_X$

(i) $\overset{\text{Thm}}{\Rightarrow}$

$$\begin{array}{ccc} (f \circ g)_* & = & (id_Y)_* : H_n(Y) \to H_n(Y) \\ \| & & \| \\ f_* \circ g_* & = & id_{H_n(Y)} \end{array}$$

(ii) $\overset{\text{Thm}}{\Rightarrow}$

$$\begin{array}{ccc} (g \circ f)_* & = & (id_X)_* : H_n(X) \to H_n(X) \\ \| & & \| \\ g_* \circ f_* & = & id_{H_n(X)} \end{array}$$

So $f_*$ is an isomorphism #

$((f_*)^{-1} = g_*)$

Example

Prove

$$H_n(\mathbb{R}^d) \cong \begin{cases} \mathbb{Z}, & n = 0 \\ 0, & n > 0. \end{cases}$$

Pf

Let

$$f : * \longrightarrow \mathbb{R}^d, \quad * \longmapsto \vec{0} \in \mathbb{R}^d$$

$$g : \mathbb{R}^d \longrightarrow *$$

Then

$$g \circ f : * \to \mathbb{R}^d \to * \quad = \quad id_*$$

and

$$f \circ g : \mathbb{R}^d \xrightarrow{g} * \xrightarrow{f} \mathbb{R}^d, \quad \vec{x} \longmapsto \vec{0}$$

Let

$$H : \mathbb{R}^d \times \overset{[0,1]}{\overset{\shortparallel}{I}} \longrightarrow \mathbb{R}^d \quad \text{— continuous}$$

$$H(\vec{x}, t) = t \cdot \vec{x}$$

$$\Rightarrow \quad H(\vec{x}, 0) = 0 \cdot \vec{x} = \vec{0} = (f \circ g)(\vec{x})$$

$$H(\vec{x}, 1) = 1 \cdot \vec{x} = \vec{x} = id_{\mathbb{R}^d}(\vec{x})$$

So

$$f \circ g \sim_H id_{\mathbb{R}^d}$$

$\Rightarrow$ $f$ is a homotopy equivalence

$$\Rightarrow \quad H_n(*) \cong H_n(\mathbb{R}^d)$$

$$\text{IIS} \quad \begin{cases} \mathbb{Z}, & n=0 \\ 0, & n>0 \end{cases} \qquad \#$$

## § Exact sequence and computation of homology

Def

A chain complex

$$\cdots \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots$$

is exact if $\ker(\partial_n) = \operatorname{im}(\partial_{n+1})$ $\forall n$.

An exact sequence of the form

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

is called a short exact sequence

i.e. $i$ is 1-1, $p$ is onto

$\operatorname{im}(i) = \ker(p)$

## Thm (Thm 2.13)

If $X$ is a space and $A$ is a nonempty closed subspace that is a deformation retract of some neighborhood in $X$, then we have the following exact seq:

$$\cdots \to \tilde{H}_n(A) \xrightarrow{i_*} \tilde{H}_n(X) \xrightarrow{q_*} \tilde{H}_n(X/A)$$
$$\xrightarrow{\partial} \tilde{H}_{n-1}(A) \xrightarrow{i_*} \cdots \to \tilde{H}_0(X/A) \to 0$$

where $i: A \hookrightarrow X$ is the inclusion map
$q: X \to X/A$ is the quotient map


## Cor (Cor 2.14)

Let $n$ be a positive integer.

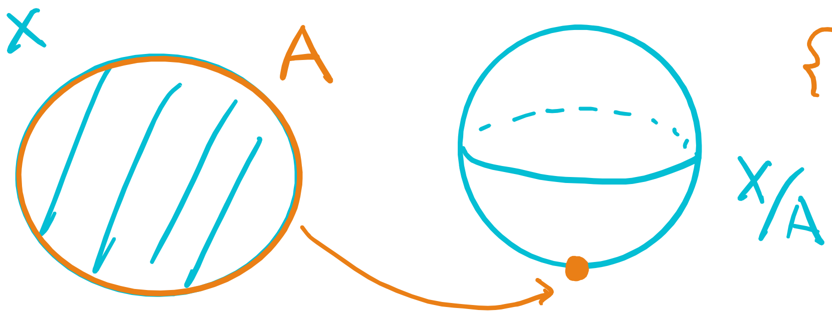
$$H_k(S^n) \cong \begin{cases} \mathbb{Z} & \text{if } k = 0, n \\ 0 & \text{if } k \neq 0, n. \end{cases}$$

pf

$\{\vec{x}\in\mathbb{R}^n \mid \|\vec{x}\|\le 1\}$

Apply Thm to $(X,A) = (\underline{D^n}, \underline{S^{n-1}})$

$\{\vec{x}\in\mathbb{R}^n \mid \|\vec{x}\|=1\}$




Note

(i) $X/A \cong S^n$

(ii) $D^n$ is contractible

$\Rightarrow \tilde{H}_k(D^n) \cong 0 \quad \forall k$

By Thm, we have the exact seq:

$$\cdots \to \tilde{H}_k(S^{n-1}) \to \boxed{\tilde{H}_k(D^n)} \to \underline{\tilde{H}_k(D^n/S^{n-1} \cong S^n)}$$

exact here

1-1

$\cong$

$$\xrightarrow{\text{onto}} \underline{\tilde{H}_{k-1}(S^{n-1})} \to \boxed{\tilde{H}_{k-1}(D^n)} \to \cdots$$

$=0$

exact here

Exactness $\Rightarrow$

$$\tilde{H}_k(S^n) \cong \tilde{H}_{k-1}(S^{n-1}) \cong \tilde{H}_{k-2}(S^{n-2}) \cong \cdots$$

$$\cdots \quad \tilde{H}_{k-n}(\underset{\text{two points}}{\underline{\underline{S^0}}}) \cong \begin{cases} \mathbb{Z} & k-n=0 \\ 0 & k-n\neq 0 \end{cases}$$

#

**Cor** ( Cor 2.15 )

The boundary $\partial D^n$ of $D^n$ is NOT a retract of $D^n$.

pf

If $\partial D^n \cong S^{n-1}$ is a retract of $D^n$, i.e.

$\exists\ r: D^n \to \partial D^n$ s.t.

$$r \circ i = \mathrm{id}_{\partial D^n} : \partial D^n \xrightarrow[\text{inclusion}]{i} D^n \xrightarrow{r} \partial D^n$$

then

$$\begin{tikzcd}
\tilde{H}_{n-1}(\partial D^n) \arrow[r, "i_*"] \arrow[rr, bend left, "\mathrm{id} = (\mathrm{id}_{\partial D^n})_*"] \arrow[d, phantom, "\cong" rotate=-90] & \tilde{H}_{n-1}(D^n) \arrow[r, "r_*"] \arrow[d, phantom, "\cong" rotate=-90] & \tilde{H}_{n-1}(\partial D^n) \arrow[d, phantom, "\cong" rotate=-90] \\
\mathbb{Z} \arrow[rr, bend right, "\text{zero map} \neq \mathrm{id}."'] & 0 & \mathbb{Z}
\end{tikzcd}$$

$(\rightarrow\leftarrow)$ #

<u>Cor</u> (Thm 1.9)

Every continuous map

$$f: D^n \to D^n$$

has a fixed point (i.e. $f(x_0)=x_0$)

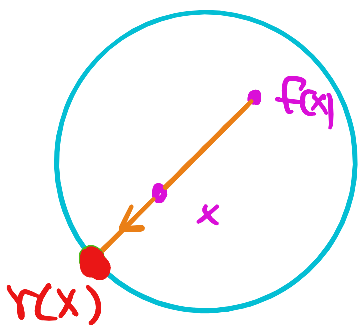
pf

Assume that $f(x) \neq x \quad \forall x \in D^n = \{x \in \mathbb{R}^n \mid \|x\| \le 1\}$

Define

$$r: D^n \to \partial D^n$$

by




One can show $r$ is continuous.

It is a retraction of $D^n$ onto $\partial D^n$ $(\rightarrow\leftarrow)$

#

## § Short exact seq. of chain complexes

Let

$$(A, \partial^A), \quad (B, \partial^B), \quad (C, \partial^C)$$

be chain complexes.

Let

$$i : (A, \partial^A) \longrightarrow (B, \partial^B)$$

$$j : (B, \partial^B) \longrightarrow (C, \partial^C)$$

be chain maps.

We say

$$0 \to (A, \partial^A) \xrightarrow{i} (B, \partial^B) \xrightarrow{j} (C, \partial^C) \to 0$$

is a short exact sequence of chain complexes if

$$0 \to A_n \xrightarrow{i_n} B_n \xrightarrow{j_n} C_n \to 0$$

is a short exact seq. for each $n$.

$$\bar{i}_*([a]) = [i_n(a)] = [b - \partial^B_{n+1}(y)] = [b]$$

$$\operatorname{im}(i_*) \subset$$

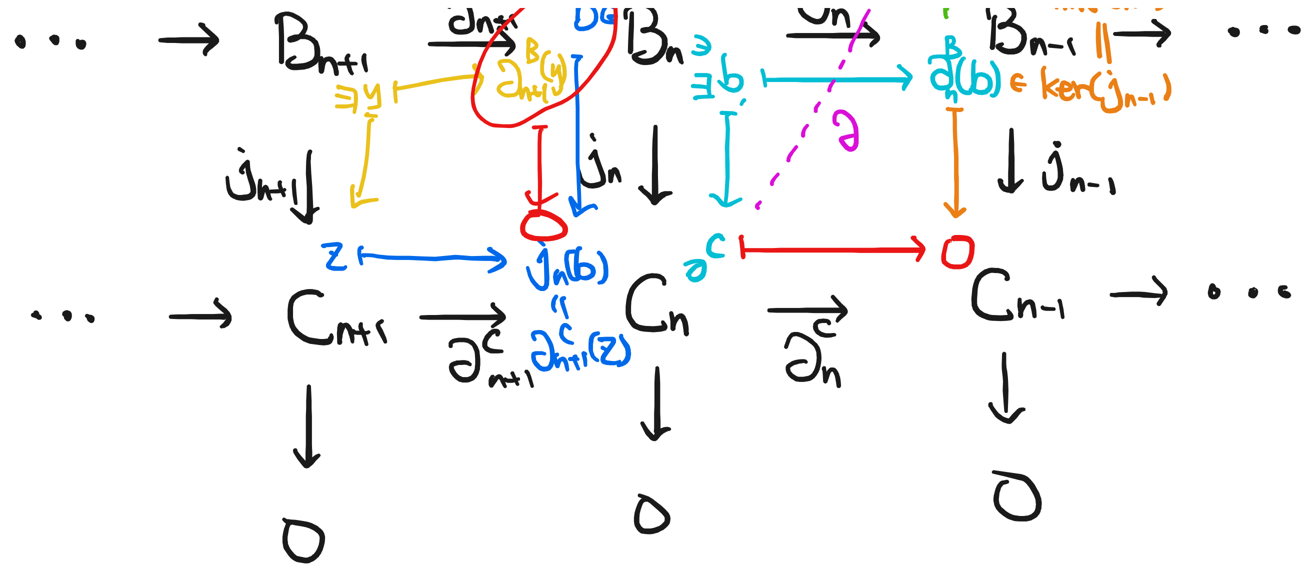
$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & A_{n+1} \arrow[r, "\partial^A_{n+1}"] \arrow[d, "i_{n+1}"] & \exists a \quad A_n \arrow[r, "\partial^A_n"] \arrow[d, "i_n"] & \exists a \in A_{n-1} \arrow[r] \arrow[d, "i_{n-1}"] & \cdots
\end{tikzcd}$$

$$b - \partial^B_{n+1}(y)$$

$$\partial([c]) := [a]$$

$$\operatorname{im}(i_{n-1})$$

$$\partial^B \qquad b \qquad \partial^B$$




<u>Thm</u>

A short exact seq. of complexes induces a long exact seq. of homology groups:

$$(*)\quad \cdots \to H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{j_*} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

where the connecting homomorphism

$$\partial : \overset{[c] \in}{H_n(C)} \longrightarrow H_{n-1}(A)$$

is defined by "diagram chasing"

Pf

① The connecting homomorphism

$$\partial : H_n(C) \longrightarrow H_n(A), \quad \partial([c]) = [a]$$

is well-defined:

(a) $a \in \ker(\partial^A_{n-1})$ ✓

(b) $[a]$ is indep. of the choice of $b$: ✓

If $b' \in B_n$ s.t. $j_n(b') = c$, then ...

(c) $[a]$ is indep. of the choice of $c$: ✓

$[c] = [c + \partial^C_{n+1}(z)]$

② $\partial$ is a group homomorphism: ✓

③ The seq. ⊛ is exact:

(a) $\mathrm{im}(i_*) \subseteq \ker(j_*)$: $\because$ $j \circ i = 0$

(b) $\ker(j_*) \subseteq \mathrm{im}(i_*)$: Let $[b] \in H_n(B)$, $j_*([b]) = 0$

$j_*([b]) = [j_n(b)]$

i.e. $j_n(b) = \partial^C_{n+1}(z)$ for some $z$

$\mathrm{im}(i_*) = \ker(j_*)$ ✓

(c) $\mathrm{im}(j_*) \subseteq \ker(\partial)$: ✓

⋮

#

## § Relative homology

If $A \subseteq X$ is a subspace of a topological sp. $X$, If $A$ is a subspace of a top[illegible]

then

$$C_n(A) = \text{free ab. gp. gen. by } \sigma : \Delta^n \to A$$

$$\wedge\!\!|$$

$$C_n(X) = \text{free ab. gp gen by } \sigma : \Delta^n \longrightarrow X$$

Let

$$C_n(X,A) = C_n(X) \big/ C_n(A)$$

Since

$$\partial^X : C_n(X) \longrightarrow C_{n-1}(X)$$

preserves the singular chains in $A$

i.e.

$$\partial^X ( C_n(A)) \subseteq C_{n-1}(A),$$

it induces a group homomorphism

$$\partial_n^{X,A} : C_n(X,A) \longrightarrow C_{n-1}(X,A)$$

$$\begin{array}{ccc} \| & & \| \\ C_n(X)/C_n(A) & & C_{n-1}(X)/C_{n-1}(A) \\ [x] & \longmapsto & [\partial_n^{X,A}(x)] \end{array}$$

Furthermore,

$$\partial^X \circ \partial^X = 0 \Rightarrow \partial^{X,A} \circ \partial^{X,A} = 0$$

So we have a chain complex

$$\cdots \to C_n(X,A) \xrightarrow{\partial_n^{X,A}} C_{n-1}(X,A) \to \cdots$$

The associated homology group

$$H_n(X,A) = \ker(\partial_n^{X,A}) / \operatorname{im}(\partial_{n+1}^{X,A})$$

is called the relative homology group

Note that

$$0 \to C_n(A) \xhookrightarrow{i} C_n(X) \xrightarrow{q} C_n(X)/C_n(A) \to 0$$
$$C_n(X)/C_n(A) = C_n(X,A)$$

is a short exact seq. of chain complexes.