Alg Topo 3/19

Recall

Continuous map

$$f : X \to Y$$

$$\rightsquigarrow \quad f_\# : (C_\bullet(X), \partial^X) \to (C_\bullet(Y), \partial^Y)$$

$$\rightsquigarrow \quad f_* : H_n(X) \longrightarrow H_n(Y) \qquad \forall n.$$

Prop (Functorial properties)

Assume $X \xrightarrow{f} Y \xrightarrow{g} Z$ are continuous maps. Then

(i) $(g \circ f)_* = g_* \circ f_*$

$$H_n(X) \xrightarrow{f_*} H_n(Y) \xrightarrow{g_*} H_n(Z), \qquad (g \circ f)_* : H_n(X) \to H_n(Z)$$

Consider

(ii) $id_X : X \to X$.

$$(id_X)_* = id_{H_n(X)}$$

$$H_n(X) \longrightarrow H_n(X).$$

Sketch of pf

$$\Delta^n \xrightarrow{\sigma} X \xrightarrow{g \circ f} Z$$

$$(g \circ f)_* ([\sigma]) = [(g \circ f) \circ \sigma]$$

$$= [g \circ (f \circ \sigma)]$$

$$= g_* ([f \circ \sigma])$$

$$= (g_* \circ f_*)([\sigma])$$

#

Cor

If X and Y are homeomorphic,

then $H_n(X) \cong H_n(Y) \quad \forall n.$

pf

X and Y are homeomorphic

$\Leftrightarrow \exists f : X \to Y$ and $g : Y \to X$

$\Leftrightarrow \exists\ f: X \to Y$ and $g: Y \to X$

s.t. (i) $f \circ g = id_Y$

(ii) $g \circ f = id_X$

Prop

$\Rightarrow$ (i) $\Rightarrow$ $(f \circ g)_* = f_* \circ g_* : H_n(Y) \xrightarrow{g_*} H_n(X) \xrightarrow{f_*} H_n(Y)$ ($id_{H_n(Y)}$)

$$\begin{array}{ccc} (f\circ g)_* & = & f_* \circ g_* \\ \| & & \| \\ (id_Y)_* & = & id_{H_n(Y)} \end{array}$$

(ii) $\Rightarrow$

$$\begin{array}{ccc} (g\circ f)_* & = & g_* \circ f_* \\ \| & & \| \\ (id_X)_* & = & id_{H_n(X)} \end{array}$$

$\Rightarrow$ $f_* : H_n(X) \to H_n(Y)$ are gp iso $\forall n$.

$\Rightarrow$ $H_n(X) \cong H_n(Y)$ $\forall n$ #

Def (p.2-4)

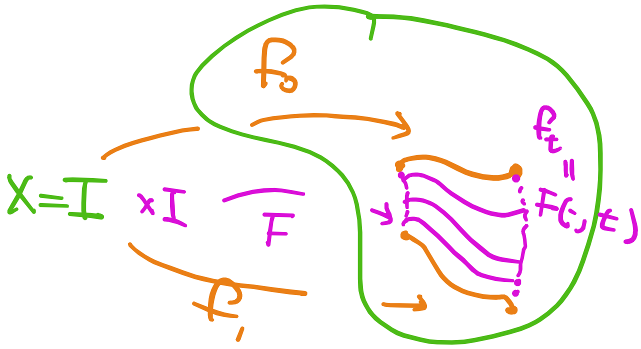
Let $f_0, f_1 : X \longrightarrow Y$ be continuous maps.

The maps f_0 and f_1 are homotopic if there exists a continuous map ($I = [0,1]$)

$$F : X \times I \longrightarrow Y$$

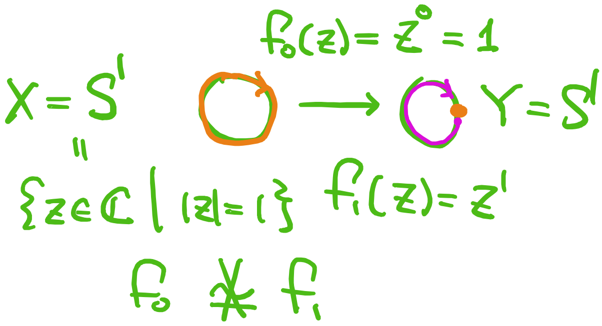
s.t.

$$f_0(x) = F(x, 0)$$



$$f_1(x) = F(x, 1)$$

$\forall\ x \in X$.



Such a F is called a homotopy from f_0 to f_1.

In this situation, we will write

$$f_0 \simeq_F f_1 \quad \text{or} \quad f_0 \simeq f_1$$

$$(\text{or} \quad f_0 \sim f_1)$$

A map $f : X \to Y$ is called a homotopy equivalence if

$\exists\ g : Y \longrightarrow X$ s.t.

"homotopy inverse"

$$g \circ f \simeq id_X$$

$$f \circ g \simeq id_Y$$

If there exists a homotopy equivalence $f : X \to Y$, we say that X and Y are homotopy equivalent or have the same homotopy type

A space is contractible if it is homotopy equivalent to a point

Let $A \subset X$. A homotopy

$$F : X \times I \to Y$$

with the property

$$F(a,t) = F(a,0) \quad \forall a \in A,\ t \in I,$$

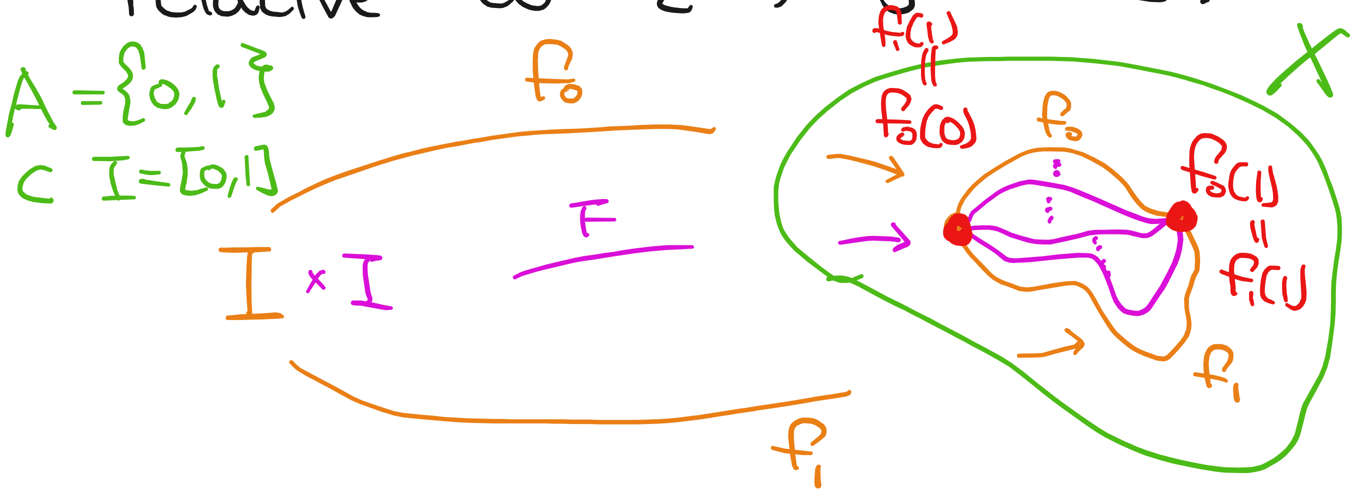
is called a homotopy relative to A.

is called a homotopy relative to A.

A path homotopy is a homotopy

$$I \times I \longrightarrow X$$

relative to $\{0, 1\} \subset I$.



A retraction (a topological version of projection ) of X onto A

is a map $r : X \to X$ s.t. $r(X) = A$ and $r(a) = a \ \forall a \in A$.

Equivalently, a retraction can be defined to be a map $r : X \to X$

s.t. $r^2 = r \circ r = r$.

(The equivalence is shown by setting $A = \mathrm{im}(r) = r(X)$)

A deformation retraction is a homotopy from id_X to a retraction r of X onto $A = \mathrm{im}(r)$.

If $\exists$ such a deformation retraction, we say that X deformation retracts to the subspace A, or that A is a deformation retract of X.

Remark

If $r: X \to X$ is a retraction of X onto A, then it can be identified

with ... $A = \mathrm{im}\, r$

with

$$\tilde{r} : X \to A, \quad \tilde{r}(x) = r(x)$$

"corestriction"

Let

$$i : A \hookrightarrow X, \quad i(a) = a$$

be the inclusion map

Note: $\tilde{r} \circ i = \mathrm{id}_A$

If X deformation retracts to A,

then

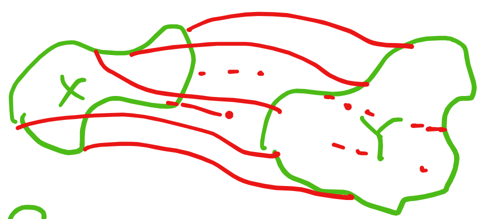
$$\mathrm{id}_X \simeq i \circ \tilde{r} \quad (= r)$$

and

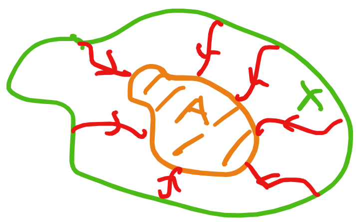
$$\mathrm{id}_A = \tilde{r} \circ i$$

exer: "$\simeq$" is an equivalence relation.

homotopy equivalence:



deformation retract:



In particular,

$$i : A \hookrightarrow X$$

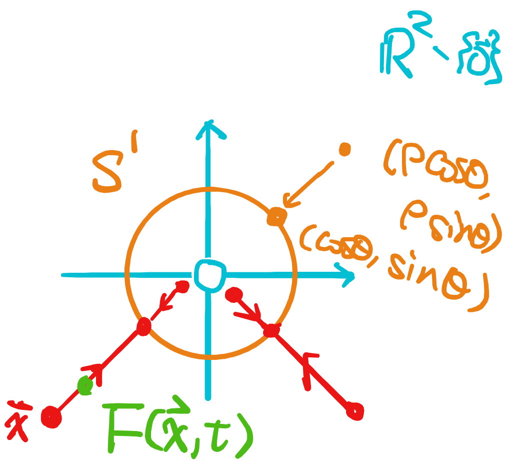
is a homotopy equivalence.

<u>Example</u>

Let

$$S^1 = \{ (x,y) \in \mathbb{R}^2 \mid x^2+y^2=1 \}$$

$$\subset \quad \mathbb{R}^2 \setminus \{(0,0)\}$$



Write

$$(x,y) = (\rho \cdot \cos\theta,\ \rho \cdot \sin\theta)$$

The map

$$r : \mathbb{R}^2 \setminus \{\vec{0}\} \xrightarrow{\tilde{r}} S^1 \xrightarrow{i} \mathbb{R}^2 \setminus \{\vec{0}\}$$

$$r(\rho \cdot \cos\theta,\ \rho \sin\theta) = (\cos\theta, \sin\theta)$$

is a retraction of $\mathbb{R}^2 \setminus \{\vec{0}\}$ onto S^1

Let

$$F : (\mathbb{R}^2 \setminus \{\vec{0}\}) \times I \longrightarrow \mathbb{R}^2 \setminus \{\vec{0}\}$$

$$F(\rho\cos\theta, \rho\sin\theta\ ;\ t)$$

$$= \Big(\big(\underbrace{(1-t)\rho + t}_{\text{the line between } \rho \text{ and } 1}\big) \cos\theta,\ \big((1-t)\rho + t\big) \cdot \sin\theta \Big)$$

$(\Rightarrow \quad F(\rho\cos\theta, \rho\sin\theta ; 0) = (\rho\cos\theta, \rho\sin\theta)$
" $id_{\mathbb{R}^2\setminus\{0\}}$ "

$F(\rho\cos\theta, \rho\sin\theta ; 1) = (\cos\theta, \sin\theta)$
" γ " $)$

This F is a deformation retraction.

In particular, the inclusion

$$S^1 \hookrightarrow \mathbb{R}^2\setminus\{\vec{0}\}$$

is a homotopy equivalence. #

<u>exer</u>

Prove that "homotopic" (relation of maps) and "homotopy equivalent" (relation of spaces) are equivalence relations.

<u>Thm</u> (Thm 2.10)

If two maps

$$f, g : X \to Y$$

are homotopic, then they induce the same homomorphisms

$$f_* = g_* : H_n(X) \longrightarrow H_n(Y), \ \forall n.$$

pf

Idea: Construct a chain homotopy

h_n = $P : C_n(X) \longrightarrow C_{n+1}(Y)$

($C_n(X)$ = C_n, $C_{n+1}(Y)$ = D_{n+1})

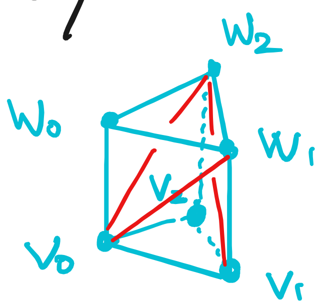
s.t.

$$\partial^Y P + P \partial^X = g_\# - f_\#$$

from a homotopy $F : X \times I \longrightarrow Y$.

Let $F : X \times I \to Y$ be a homotopy from f to g.

Consider $\Delta^n \times I$. Divide $\Delta^n \times I$ in the following way:



Define

$$P : C_n(X) \longrightarrow C_{n+1}(Y)$$

by ($\sigma : \Delta^n \longrightarrow X$ continuous map)

$$P(\sigma) = \sum_{i=0}^{n} (-1)^i F \circ (\sigma \times id_I)\Big|_{[v_0, \cdots, v_i, w_i, \cdots, w_n]}$$

where

$$\{(t_0, \cdots, t_{n+1}) \in \mathbb{R}^{n+2} \mid \sum_{i=0}^{n+1} t_i = 1, \ t_i \geq 0\}$$

$$(\sigma \times id_I)\Big|_{[v_0, \cdots, v_i, w_i \cdots w_n]} : \Delta^{n+1} \longrightarrow \Delta^n \times I \longrightarrow X \times I$$

$$(t_0, \cdots, t_{n+1}) \mapsto (t_0, \cdots, t_i, 0, \cdots, 0 ; 0) + (0, \cdots, 0, t_{i+1}, \cdots, t_n ; t_{i+1} + \cdots + t_{n+1})$$

with $(t_0, \cdots, t_{n+1}) \in \Delta^{n+1}$ and the image in $\Delta^n \times I$,

$$\mapsto \big(\sigma(t_0, \cdots, t_{i-1}, t_i + t_{i+1}, t_{i+2}, \cdots, t_n) ; t_{i+1} + \cdots + t_{n+1}\big)$$

$\in X \times I$

That is,

$P(\sigma)(t_0, \cdots, t_{n+1})$ $\quad \Delta^{n+1} \longrightarrow Y$

$$= \sum_{i=0}^{n} (-1)^i \cdot F\left(\sigma(t_0, \cdots, t_{i-1}, t_i + t_{i+1}, t_{i+2}, \cdots, t_n), t_{i+1} + \cdots + t_{n+1}\right)$$

Now we need to verify:

$$\underset{①}{\partial P} + \underset{②}{P \partial} = g_\# - f_\#$$

Recall

$$\partial(\sigma) = \sum_{i=0}^{n} (-1)^i \sigma|_{[v_0 \cdots \hat{v}_i \cdots v_n]}$$

$\sigma : \Delta^n \to X$

$\Rightarrow P(\sigma) \in C_{n+1}(Y)$

$\Rightarrow \partial P(\sigma) \in C_n(Y)$

① $(\partial P)(\sigma)(t_0, \cdots, t_n)$

$$= \sum_{i=0}^{n} (-1)^i (P\sigma)|_{[v_0 \cdots \hat{v}_i \cdots v_n]}(t_0, \cdots, t_n)$$

$$= \sum_{i=0}^{n} (-1)^i (P\sigma)(t_0, \cdots, t_{i-1}, 0, t_i, \cdots, t_n)$$

$$= \sum_{i=0}^{n} (-1)^i \Bigg(\sum_{j=0}^{i-2} (-1)^j F\left(\sigma(t_0, \cdots, t_j + t_{j+1}, \cdots t_{i-1}, 0, \cdots t_n), t_{j+1} + \cdots t_n\right)$$

$$+ (-1)^{i-1} F\left(\sigma(t_0, \cdots t_n), t_i + \cdots + t_n\right) \quad \text{“} j = i-1 \text{”}$$

$$+ (-1)^i F\left(\sigma(t_0, \cdots, t_n), t_i + \cdots + t_n\right) \quad \text{“} j = i \text{”}$$

$$+ \sum_{j=i+2}^{n-1} (-1)^{j+1} F\left(\sigma(t_0 \cdots, 0, t_{i+1}, \cdots, t_j + t_{j+1}, \cdots t_n), t_{j+1} + \cdots + t_n\right) \Bigg)$$

vanishes if $i=0$ (the $j = i-1$ term)

vanishes if $i=n$ (the $j = i$ term)

② $(P\partial)(\sigma)(t_0,\cdots,t_n)$

$$= \sum_{j=0}^{n-1} (-1)^j F\big((\partial\sigma)(t_0,\cdots,t_j+t_{j+1},\cdots t_n),\ t_{j+1}+\cdots+t_n\big)$$

$$= \sum_{j=0}^{n-1} (-1)^j \left(\sum_{i=0}^{j} (-1)^i F(\sigma(t_0,\cdots,0,t_i,\cdots,t_j+t_{j+1},\cdots t_n), t_{j+1}+\cdots+t_n) + \sum_{i=j+2}^{n} (-1)^{i+1} F(\sigma(t_0\cdots t_j+t_{j+1}\cdots 0\cdots), \cdots) \right)$$

So $\big((\partial P+P\partial)(\sigma)\big)(t_0,\cdots t_n)$

$$= F(\sigma(t_0,\cdots,t_n), \overbrace{t_0+\cdots+t_n}^{1}) - F(\sigma(t_0,\cdots t_n), 0)$$

double check

$$= g(\sigma(t_0,\cdots,t_n)) - f(\sigma(t_0,\cdots,t_n))$$

$$= (g_\#(\sigma) - f_\#(\sigma))(t_0,\cdots,t_n)$$

$\Rightarrow \quad \partial P + P\partial = g_\# - f_\#$

$\Rightarrow \quad g_* = f_*$ #