Alg Topo 3/12

<u>Prop</u>

If X is nonempty path-connected, then

$$H_0(X) \cong \mathbb{Z}.$$

<u>pf</u>

Recall that we have

$$\cdots \to C_n(X) \to C_{n-1}(X) \to \cdots \to C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

where

$$\varepsilon\left(\sum_{k=1}^{p} m_k \cdot x_k\right) = \sum_{k=1}^{p} m_k$$

Since X is nonempty, the homomorphism

$\boxed{\varepsilon \text{ is surjective.}}$

$\Rightarrow$ ③ $\bar{\varepsilon}$ is onto

Since $\varepsilon \circ \partial_1 = 0$,

$\boxed{\operatorname{im} \partial_1 \subseteq \ker \varepsilon}$

<u>Want</u>

$$H_0(X) = C_0(X) / \operatorname{im} \partial_1$$

$$\bar{\varepsilon} \ \cong \big\downarrow \qquad [\sigma] \mapsto$$

$$\mathbb{Z} \ni \varepsilon(\sigma)$$

$\Rightarrow$ ① $\bar{\varepsilon}$ is well-defined

Since ε is a group homomorphism,

$\bar{\varepsilon}$ is also a ② group homomorphism:

$$\bar{\varepsilon}([\sigma_1]+[\sigma_2]) = \bar{\varepsilon}([\sigma_1+\sigma_2]) = \varepsilon(\sigma_1+\sigma_2)$$

$$= \varepsilon(\sigma_1) + \varepsilon(\sigma_2) = \bar{\varepsilon}([\sigma_1]) + \bar{\varepsilon}([\sigma_2])$$

It remains to show $\bar{\varepsilon}$ is 1-1:

Assume $\sum_{k=1}^{p} m_k x_k \in C_0(X)$ with

$$\bar{\varepsilon}\left(\left[\sum_{k=1}^{p} m_k x_k\right]\right) = 0$$

$$\|$$

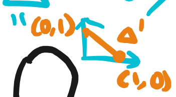
$$\varepsilon\left(\sum_{k=1}^{p} m_k x_k\right) = \sum_{k=1}^{p} m_k = 0.$$

Q: $\sum_{k=1}^{p} m_k x_k \in \operatorname{im}(\partial_1)$?

$\partial_1(\sigma) = \sigma(0,1) - \sigma(1,0)$

where

$\sigma: \Delta^1 \to X$



Then

$$\sum_{k=1}^{p} m_k \cdot x_k = m_1 \cdot \underbrace{(x_1 - x_2)}_{\partial_1(\sigma_1)} + (m_1+m_2)\underbrace{(x_2 - x_3)}_{\partial_1(\sigma_2)}$$

$$+ (m_1+m_2+m_3)(x_3 - x_4) + \cdots$$

$$+ (m_1 + \cdots + m_{p-1})(x_{p-1} - x_p)$$

$$+ \underbrace{(m_1 + \cdots + m_{p-1} + m_p)}_{\overset{\parallel}{0}\ \text{by assumption}} x_p$$

Since X is path-connected,

$\exists\ \sigma_k : \Delta^1 \longrightarrow X$ s.t.

$$\sigma_k(1,0) = x_{k+1}, \quad \sigma_k(0,1) = x_k$$

$k = 1, \cdots, p-1$

So $\partial_1(\sigma_k) = x_k - x_{k+1}$, $k = 1, \cdots, p-1$

$$\Rightarrow \sum_{k=1}^{p} m_k \cdot x_k = m_1 \cdot \partial_1(\sigma_1) + (m_1 + m_2) \cdot \partial_1(\sigma_2)$$

$$+ (m_1 + m_2 + m_3) \cdot \partial_1(\sigma_3) + \cdots$$

$$+ (m_1 + \cdots + m_{p-1}) \cdot \partial_1(\sigma_{p-1})$$

$$= \partial_1 \left(m_1 \sigma_1 + (m_1 + m_2) \sigma_2 + \cdots + (m_1 + \cdots + m_{p-1}) \sigma_{p-1} \right)$$

$$\in \operatorname{im} \partial_1$$

So

$$\bar{\varepsilon}\left(\left[\sum_{k=1}^{p} m_k \cdot x_k\right]\right) = 0$$

$$\Rightarrow \quad \sum_{k=1}^{p} m_k \cdot x_k \in \operatorname{im} \partial_1$$

$$\Rightarrow \quad \left[\sum_{k=1}^{p} m_k \cdot x_k\right] = 0 \in C_0(X) / \operatorname{im}\partial_1 = H_0(X)$$

④

$$\Rightarrow \quad \bar{\varepsilon} : H_0(X) \longrightarrow \mathbb{Z} \quad \text{is } 1\text{-}1$$

①~④ $\Rightarrow$ $\bar{\varepsilon}$ is a group isomorphism

$$H_0(X) \cong \mathbb{Z} \qquad \#$$

Some homological alg

A chain complex is a seq. of abelian groups (in general, objects in an abelian category, such as vector spaces, R-module) vector spaces, R-modules)

C_n together with a seq. of group homomorphisms.

$$\partial_n : C_n \to C_{n-1}$$

s.t.

$$\partial_n \circ \partial_{n+1} = 0 \quad \forall n$$

Usually denoted by a diagram:

$$\cdots \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots$$

Equivalently, one can consider a chain complex as an abelian gp $C_\bullet$ with a decomposition

$$C_\bullet = \bigoplus_{n \in \mathbb{Z}} C_n$$

with a gp homomorphism (of deg. $\underline{\underline{-1}}$)

$$\partial : C_\bullet \to C_{\bullet - 1}$$

s.t.

$$\partial \circ \partial = 0$$

The n-th homology of a chain complex $(C_\bullet, \partial)$ is

$$H_n(C_\bullet) = \ker(\partial_n) / \operatorname{im}(\partial_{n+1})$$

Example

Simplicial chain complex and singular chain complex of a space are chain complexes

Def

Let $(C_\bullet, \partial)$ and $(D_\bullet, \partial')$ be chain complexes. A chain map $\Phi : (C_\bullet, \partial) \to (D_\bullet, \partial)$ is a seq. of gp homomorphisms

$$\Phi_n : C_n \to D_n$$

s.t.

$$\boxed{\Phi_{n-1} \circ \partial_n} = \boxed{\partial'_n \circ \Phi_n} \qquad \forall n$$

That is, the following diagram

Commutes:

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "\Phi_{n+1}"'] \arrow[dr, phantom, "\Phi_n\circ\partial_{n+1} = \partial'_{n+1}\circ\Phi_{n+1}"] & C_n \arrow[r, "\partial_n"] \arrow[d, "\Phi_n"] \arrow[dr, phantom, "="] & C_{n-1} \arrow[r, "\partial_{n-1}"] \arrow[d, "\Phi_{n-1}"] & \cdots \\
\cdots \arrow[r] & D_{n+1} \arrow[r, "\partial'_{n+1}"'] & D_n \arrow[r, "\partial'_n"'] & D_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

Prop (Prop 2.9)

A chain map

$$\Phi : (C_\bullet, \partial) \to (D_\bullet, \partial')$$

induces group homomorphisms of homology

groups:

$$\Phi_* : \underbrace{H_n(C_\bullet)}_{\substack{\| \\ \ker\partial_n / \operatorname{im}\partial_{n+1}}} \longrightarrow \underbrace{H_n(D_\bullet)}_{\substack{\| \\ \ker\partial'_n / \operatorname{im}\partial'_{n+1}}}$$

$$\Phi_*([c]) = [\Phi(c)]$$

pf

① Φ_* is well-defined : in $H_n(C_\bullet) = \ker\partial_n / \operatorname{im}\partial_{n+1}$

Assume $c, c' \in C_n$ s.t. $[c] = [c']$

i.e. $\partial_n(c) = \partial_n(c') = 0$ and

$\exists\, x \in C_{n+1}$ s.t.

$$c' = c + \partial_{n+1}(x)$$

Check: $\underset{[\Phi(c)]\,=}{\Phi_*([c])} = \underset{[\Phi(c')]\,=}{\Phi_*([c'])}$

in $H_n(D_\bullet) = \ker\partial'_n / \operatorname{im}\partial'_{n+1}$

(1-a) $\Phi_n(c)$, $\Phi_n(c') \overset{?}{\in} \ker\partial'_n$

$$\partial'_n(\Phi_n(c)) = \Phi_{n-1}(\underset{\substack{\| \\ 0}}{\partial_n(c)}) = 0$$

by def. of chain map:

$\partial'_n \circ \Phi_n = \Phi_{n-1} \circ \partial_n$

Similarly, $\partial_n'(\Phi_n(c')) = 0$

That is,

$$\Phi_n(c),\ \Phi_n(c') \in \ker(\partial_n')$$

$\Rightarrow$ we have

$$[\Phi_n(c)],\ [\Phi_n(c')] \in H_n(D_\cdot) = \frac{\ker(\partial_n')}{\operatorname{im}(\partial_n')}$$

(1-b) $[\Phi_n(c)] \overset{?}{=} [\Phi_n(c')]$

By assumption, $c' = c + \partial_{n+1}(x)$.

$$\Rightarrow\ [\Phi_n(c')] = [\Phi_n(c + \partial_{n+1}(x))]$$

$$= [\Phi_n(c) + \underbrace{\Phi_n \partial_{n+1}}_{= \partial_{n+1}' \Phi_{n+1}}(x)]$$

$$= [\Phi_n(c)] + \underbrace{[\partial_{n+1}'(\boxed{\Phi_{n+1}(x)})]}_{= 0 \text{ in } H_n(D_\cdot) = \frac{\ker \partial_n'}{\operatorname{im} \partial_{n+1}'}}$$

$\operatorname{im}(\partial_{n+1}')$

$$= [\Phi_n(c)]$$

$\Rightarrow \quad \Phi_*([c]) = \Phi_*([c'])$ in $H_n(D_\bullet)$ if $[c]=[c']$ in $H_n(C_\bullet)$

② Φ_* is a gp homomorphism:

$$\Phi_*([c]+[c']) = \Phi_*([c+c'])$$

$$= [\Phi(c+c')]$$

$$= [\Phi(c) + \Phi(c')]$$

$$= [\Phi(c)] + [\Phi(c')]$$

$$= \Phi_*([c]) + \Phi_*([c']) \quad \#$$

<u>Composition of chain maps:</u>

Assume $\Phi : (C_\bullet, \partial) \to (D_\bullet, \partial')$

$\Psi : (D_\bullet, \partial') \to (E_\bullet, \partial'')$

are chain maps.

Then

$\Psi\circ\Phi : (C_\bullet, \partial) \longrightarrow (E_\bullet, \partial'')$

$\Psi \circ \Phi : (C_\bullet, \partial) \longrightarrow (E_\bullet, \partial'')$

is also a chain map.

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "\Phi_{n+1}"'] & C_n \arrow[r, "\partial_n"] \arrow[d, "\Phi_n"] & C_{n-1} \arrow[r] \arrow[d, "\Phi_{n-1}"] & \cdots \\
\cdots \arrow[r] & D_{n+1} \arrow[r, "\partial'_{n+1}"] \arrow[d, "\Psi_{n+1}"'] & D_n \arrow[r, "\partial'_n"] \arrow[d, "\Psi_n"] & D_{n-1} \arrow[r] \arrow[d, "\Psi_{n-1}"] & \cdots \\
\cdots \arrow[r] & E_{n+1} \arrow[r, "\partial''_{n+1}"'] & E_n \arrow[r, "\partial''_n"'] & E_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

$$\partial''_n \circ (\Psi_n \circ \Phi_n) = (\partial''_n \circ \Psi_n) \circ \Phi_n$$

$$= (\Psi_{n-1} \circ \partial'_n) \circ \Phi_n = \Psi_{n-1} \circ (\partial'_n \circ \Phi_n)$$

$$= \Psi_{n-1} \circ (\Phi_{n-1} \circ \partial_n) = (\Psi_{n-1} \circ \Phi_{n-1}) \circ \partial_n$$

Prop

① $id_* = id_{H_n(C_\bullet)}$

"functorial properties"

$(\Psi \circ \Phi)_* : H_n(C_\bullet) \longrightarrow H_n(E_\bullet)$

② $\quad \|$

$\Psi_* \circ \Phi_* : H_n(C_\bullet) \to H_n(D_\bullet) \to H_n(E_\bullet)$

pf

$(\Psi \circ \Phi)_*([c]) = [\Psi(\Phi(c))]$

$\forall [c] \in H_n(C_\bullet)$

$$= \Psi_*([\Phi(c)])$$
$$= \Psi_*(\Phi_*([c])) \qquad \#$$

<u>Def</u>

Let $\Phi, \Psi : (C_\bullet, \partial) \to (D_\bullet, \partial')$ be chain maps. We say Φ are Ψ are (chain) homotopic if there exists a seq. of gp. homo.

$$h_n : C_n \to D_{n+1}$$

s.t.

$$\boxed{\Phi_n - \Psi_n = \partial'_{n+1} \circ h_n + h_{n-1} \circ \partial_n .} \quad (*)$$

Such a seq. $h = (h_n)$ is called a (chain) homotopy.

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[rr, "\partial_{n+1}"] \arrow[d, shift right, "\Phi_{n+1}"'] \arrow[d, shift left, "\Psi_{n+1}"] & & C_n \arrow[rr, "\partial_n"] \arrow[dll, "h_n"'] \arrow[d, shift right, "\Phi_n"'] \arrow[d, shift left, "\Psi_n"] & & C_{n-1} \arrow[r] \arrow[dll, "h_{n-1}"] \arrow[d, shift right, "\Phi_{n-1}"'] \arrow[d, shift left, "\Psi_{n-1}"] & \cdots \\
 & {} & & {} & & {} &
\end{tikzcd}$$

$$\cdots \to D_{n+1} \xrightarrow[\partial'_{n+1}]{} D_n \xrightarrow[\partial'_n]{} D_{n-1} \to \cdots$$

<u>Prop</u>

If two chain maps

$$\Phi, \Psi : (C_\bullet, \partial) \to (D_\bullet, \partial')$$

are chain homotopic, then

$$\Phi_* = \Psi_* : H_n(C_\bullet) \to H_n(D_\bullet) \quad \forall n.$$

<u>pf</u>

Suppose $h = (h_n)$ is a chain homotopy.

Given any $[x] \in H_n(C_\bullet) = \ker\partial_n / \operatorname{im}\partial_{n+1}$,

we have $\partial_n(x) = 0$.

$\Rightarrow$

$$\Phi_*([x]) - \Psi_*([x])$$

$$= [\Phi_n(x) - \Psi_n(x)]$$

$$\overset{(*)}{=} [\partial'_{n+1} h_n(x) + h_n \cancel{\partial_n(x)}^{=0}]$$

$$= [\underbrace{\partial'_{n+1}(h_n(x))}_{\in \, im(\partial'_{n+1})}] = 0 \text{ in } H_n(D_\bullet)$$

So $\Phi_* = \Psi_*$ #

§ Induced homomorphisms on singular homology and homotopy invariance

Let $f : X \to Y$ be a continuous map between topological spaces.

Given any singular n-simplex

$$\sigma : \Delta^n \longrightarrow X$$ ← a continuous map

in X, we have an induced singular n-simplex in Y:

$$f \circ \sigma : \Delta^n \xrightarrow{f \circ \sigma} Y$$

Define

$$f_\# : C_n(X) \longrightarrow C_n(Y)$$

n-simplex in Y

$$f_\#\Big(\sum_k m_k \cdot \sigma_k \Big) = \sum_k m_k \cdot (f \circ \sigma_k)$$

n-simplex in X

which is a group homomorphism

<u>Lemma</u>

$$f_\# : (C_\bullet(X), \partial^X) \longrightarrow (C_\bullet(Y), \partial^Y)$$

is a chain map, i.e., the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1}(X) \arrow[r, "\partial^X_{n+1}"] \arrow[d, "f_\#"'] & C_n(X) \arrow[r, "\partial^X_n"] \arrow[d, "f_\#"'] & C_{n-1}(X) \arrow[r] \arrow[d, "f_\#"] & \cdots \\
\cdots \arrow[r] & C_{n+1}(Y) \arrow[r, "\partial^Y_{n+1}"'] & C_n(Y) \arrow[r, "\partial^Y_n"'] & C_{n-1}(Y) \arrow[r] & \cdots
\end{tikzcd}$$

Commutes.

<u>pf</u>

$$f_\# \partial_n^X(\sigma) = f_\#\left(\sum_{i=0} (-1)^i \sigma|_{[v_0 \cdots \hat{v}_i \cdots v_n]} \right)$$

$$= \sum_{i=0}^{n} (-1)^i f_\#(\sigma|_{[v_0 \cdots \hat{v}_i \cdots v_n]})$$

$$= \sum_{i=0}^{n} (-1)^i (f \circ \sigma)|_{[v_0 \cdots \hat{v}_i \cdots v_n]}$$

$$= \partial_n^Y (f \circ \sigma)$$

Recall:

$$\partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i \sigma|_{[\cdots \hat{v}_i \cdots]}$$

$$= \partial_n^Y(f_\#(\sigma))$$

$$\Rightarrow \quad f_\# \circ \partial_n^X = \partial_n^Y \circ f_\# \qquad \#$$

As a consequence, a continuous map $f: X \to Y$ induces a group homo.

$$f_*: \underset{\ker(\partial_n^X)/\mathrm{im}(\partial_{n+1}^X)}{\overset{\parallel}{H_n(X)}} \longrightarrow \underset{\ker(\partial_n^Y)/\mathrm{im}(\partial_{n+1}^Y)}{\overset{\parallel}{H_n(Y)}} \quad \forall n$$

Remark

Since the diagram

$$
\begin{tikzcd}
\cdots \arrow[r] & C_2(X) \arrow[r, "\partial_2^X"] \arrow[d, "f_\#"'] & C_1(X) \arrow[r, "\partial_1^X"] \arrow[d, "f_\#"'] & C_0(X) \arrow[r, "\varepsilon"] \arrow[d, "f_\#"'] \arrow[dr, phantom, "\text{exer}"] & \mathbb{Z} \arrow[d, "id_{\mathbb{Z}}"] \\
\cdots \arrow[r] & C_2(Y) \arrow[r, "\partial_2^Y"'] & C_1(Y) \arrow[r, "\partial_1^Y"'] & C_0(Y) \arrow[r, "\varepsilon"'] & \mathbb{Z}
\end{tikzcd}
$$

commutes, we also have the induced homomorphisms between the reduced homology groups:

$$f_* : \tilde{H}_n(X) \longrightarrow \tilde{H}_n(Y) \quad \forall n.$$