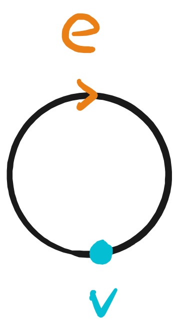
Alg. Topo. 3/5

Example

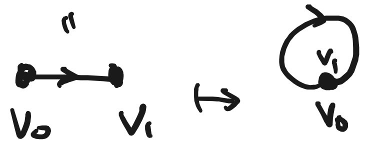
$X = S^1 =$ 

Δ-complex: $\{*\} = \Delta^0$

$\sigma_0 : \Delta^0 \longrightarrow S^1$

$\quad * \longmapsto v$

$\sigma_1 : \Delta^1 \longrightarrow S^1$



$\rightsquigarrow$ Simplicial chain complex:

$$0 \xrightarrow{\partial_2} \underset{\mathbb{Z}}{\Delta_1(S^1)} \xrightarrow{\partial_1} \underset{\mathbb{Z}}{\Delta_0(S^1)} \xrightarrow{\partial_0} 0$$

$(a \in \mathbb{Z})$ $\quad a\,\sigma_1 \longmapsto a(\sigma_0 - \sigma_0) = 0$

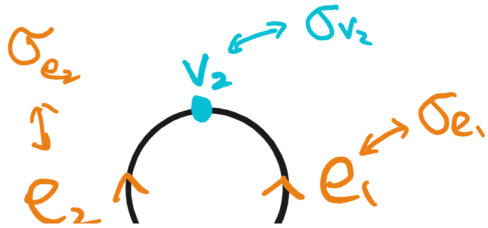
So

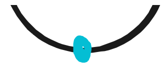
$$H_0^{\Delta}(S^1) = \ker\partial_0 / \operatorname{im}\partial_1 \cong \mathbb{Z}/0 = \mathbb{Z}$$

$$H_1^{\Delta}(S^1) = \ker\partial_1 / \operatorname{im}\partial_2 \cong \mathbb{Z}/0 = \mathbb{Z}$$

$$H_k^{\Delta}(S^1) = 0 \quad \forall\, k \geq 2.$$

Another Δ-complex structure on S^1:





$v_1 \leftrightarrow \sigma_{v_1}$

$$0 \to \Delta_1(S^1) = \mathbb{Z}\sigma_{e_1} \oplus \mathbb{Z}\sigma_{e_2} \xrightarrow{\partial_1} \Delta_0(S^1) = \mathbb{Z}\sigma_{v_1} \oplus \mathbb{Z}\sigma_{v_2} \xrightarrow{\partial_0} 0$$

$$a\sigma_{e_1} + b\sigma_{e_2} \longmapsto a(\sigma_{v_2} - \sigma_{v_1}) + b(\sigma_{v_2} - \sigma_{v_1}) = (a+b)(\sigma_{v_2} - \sigma_{v_1})$$

$$\Rightarrow H_0^{\Delta}(S^1) = \ker\partial_0 / \operatorname{im}\partial_1 \cong \mathbb{Z}\oplus\mathbb{Z} / \langle(-1,1)\rangle \cong \mathbb{Z}$$

$$\mathbb{Z} \ni x \mapsto x + \langle(-1,1)\rangle$$

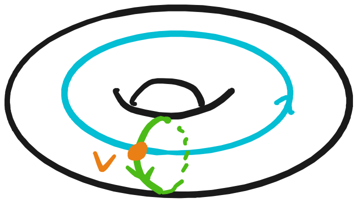
$$H_1^{\Delta}(S^1) = \ker\partial_1 / \operatorname{im}\partial_2 = \{(a,-a) \in \mathbb{Z}\oplus\mathbb{Z} \mid a \in \mathbb{Z}\} / 0 \cong \mathbb{Z}$$

$$H_k^{\Delta}(S^1) = 0 \quad \forall\, k \geq 2$$

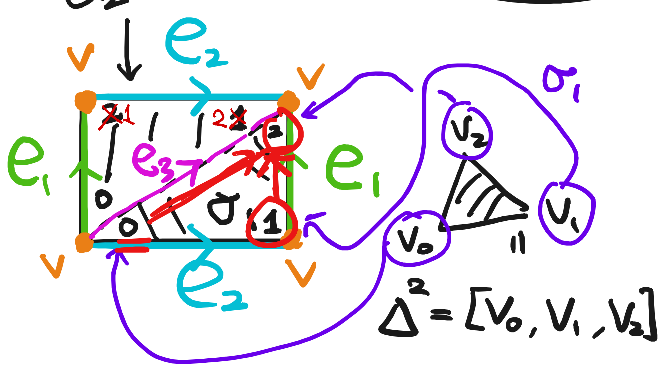
#

Example

Consider torus

$T = S^1 \times S^1 =$ 

$\partial_2(\sigma) = \sigma|_{[v_1, v_2]} - \sigma|_{[v_0, v_2]} + \sigma|_{[v_0, v_1]}$

$=$ 

$0 \xrightarrow{\partial_3} \Delta_2(T) = \mathbb{Z}\sigma_1 \oplus \mathbb{Z}\sigma_2$ $\quad (a+b)(e_1+e_2-e_3)$

$a\sigma_1 + b\sigma_2 \mapsto a(e_1 - e_3 + e_2) \; \|$

$+ b(+e_2 \overset{-e_3}{\cancel{-e_1}} + \overset{+e_1}{\cancel{e_3}})$

$\xrightarrow{\partial_2} \Delta_1(T) = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \mathbb{Z}e_3$

$ae_1 + be_2 + ce_3$

$\mapsto a(v-v) + b(v-v) + c(v-v) = 0$

$\xrightarrow{\partial_1} \Delta_0(T) = \mathbb{Z}v \to 0$

So $H_0^{\Delta}(T) \cong \mathbb{Z}$

$H_1^{\Delta}(T) = \ker(\partial_1) / \mathrm{im}(\partial_2) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} / \langle (1,1,-1) \rangle$

(x, y, z)

$(x+z, y+z)$

$$\cong \mathbb{Z} \oplus \mathbb{Z}$$

$$H_2^{\Delta}(T) = \ker(\partial_2) / \operatorname{im}(\partial_3) \cong \langle (1,1) \rangle / 0 \qquad \langle (1,1) \rangle \subseteq \mathbb{Z} \oplus \mathbb{Z}$$

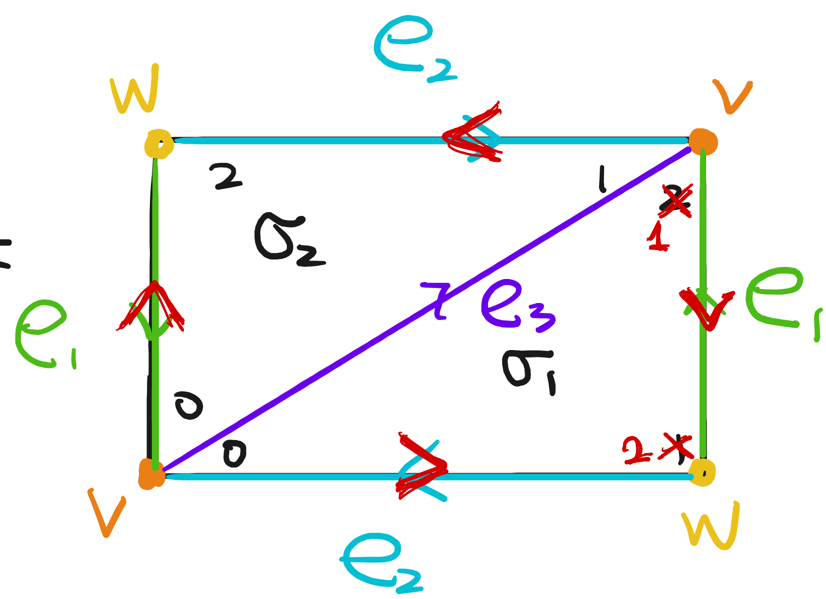
$$\cong \mathbb{Z}$$

$$H_k^{\Delta}(T) = 0 \quad \forall k \geq 3 \qquad \#$$

Example

Consider

$\mathbb{RP}^2 =$



$$0 \xrightarrow{\partial_3} \Delta_2(\mathbb{RP}^2) = \mathbb{Z}\sigma_1 \oplus \mathbb{Z}\sigma_2$$

$$a\sigma_1 + b\sigma_2$$

$$\mapsto a(e_1 - e_2 + e_3) + b(+e_2 - e_1 + e_3)$$

$$= (a-b)(e_1 - e_2 + e_3) + 2b e_3$$

$$\xrightarrow{\partial_2} \Delta_1(\mathbb{RP}^2) = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \mathbb{Z}e_3$$

$$ae_1 + be_2 + ce_3$$

$$\mapsto a(\overset{w-v}{\cancel{v-w}}) + b(\overset{w-v}{\cancel{v-w}}) + c(v-v)$$

$$= (a+b)(\overset{w-v}{\cancel{v-w}})$$

$$\xrightarrow{\partial_1} \Delta_0(\mathbb{RP}^2) = \mathbb{Z}v \oplus \mathbb{Z}w \xrightarrow{\partial_0} 0$$

So

$$H_0^{\Delta}(\mathbb{RP}^2) = \ker\partial_0 / \operatorname{im}\partial_1 \cong \mathbb{Z}\oplus\mathbb{Z} / \langle(-1,+1)\rangle$$

$$\cong \mathbb{Z}$$

$$H_1^{\Delta}(\mathbb{RP}^2) = \ker\partial_1 / \operatorname{im}\partial_2$$

$$\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}$$

$$(1,-1,+1) \quad , (0,0,1)\, c_1 \qquad \boxed{[0,0,1]}$$

$$\cong \frac{\langle(1,-1,0),\ (0,0,1)\rangle}{\langle(1,-1,+1),\ (0,0,2)\rangle}$$

[1]

$$\cong \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$$

$$H_2^{\Delta}(\mathbb{RP}^2) = \ker\partial_2 / \operatorname{im}\partial_3 = 0$$

$$H_k^{\Delta}(\mathbb{RP}^2) = 0 \quad \forall k \geq 3$$

#

Remark

Tranditionally, simplicial homology is defined for simplicial complexes which are the Δ-complexes whose simplexes are uniquely determined by their vertexes.

It can be shown that every Δ-complex can be subdivided to be a simplicial complex.

In particular, every Δ-complex

is homeomorphic to a simplicial complex.

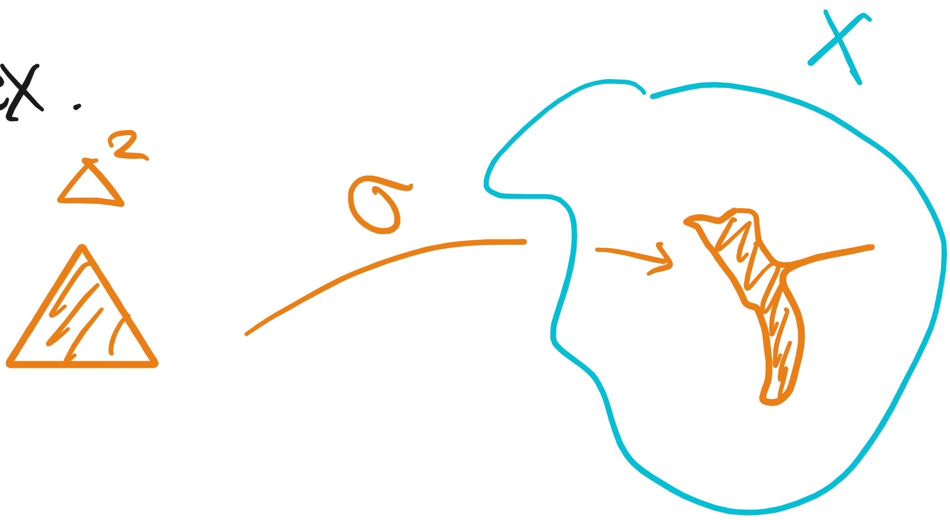
Singular homology

Let X be a topological space.

A (singular) n-simplex in X is a continuous map

$$\sigma : \Delta^n \longrightarrow X,$$

where Δ^n is the standard n-simplex.



Define

$C_n(X)$ = free abelian group freely generated by singular n-simplexes in X.

Elements in $C_n(X)$, called (singular) n-chains, can be written as finite formal sums

$$\sum_{\substack{\sigma:\Delta^n\to X \\ \text{continuous}}} m_\sigma \cdot \sigma, \qquad m_\sigma \in \mathbb{Z}.$$

which are subject to the rules:

(i) $m_\sigma = 0$ except finite $\sigma:\Delta^n \to X$;

(ii) $\sum_\sigma m_\sigma \cdot \sigma = \sum_\sigma m_\sigma' \cdot \sigma$

$$\iff m_\sigma = m_\sigma' \quad \forall \sigma:\Delta^n\to X$$

(iii) $\sum_\sigma m_\sigma \cdot \sigma + \sum_\sigma m_\sigma' \cdot \sigma$

$$= \sum (m_\sigma + m_\sigma') \cdot \sigma$$

$-\sum_{\sigma}(m_\sigma + m_\sigma') \sigma$

The <u>boundary operator</u>

$$\partial_n : C_n(X) \longrightarrow C_{n-1}(X)$$

is defined as

$$\partial_n\Big(\sum_{\sigma} m_\sigma \cdot \sigma\Big) = \sum_{\sigma} m_\sigma \cdot \partial_n(\sigma)$$

where

$$\partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i \, \sigma|_{[v_0, \cdots \hat{v}_i \cdots v_n]}$$

Here $\sigma|_{[v_0,\cdots \hat{v}_i \cdots v_n]} : \Delta^{n-1} \longrightarrow X$ is the $(n-1)$-simplex

$$\Delta^{n-1} = \Big\{ (t_0, \cdots, t_{n-1}) \in \mathbb{R}^n \;\Big|\; \sum_{i=0}^{n-1} t_i = 1,\ t_i \geq 0 \Big\}$$

$$\sigma : \Delta^n \subseteq \mathbb{R}^{n+1} \to X \qquad \Delta^n = [v_0, \cdots, v_n] = \{ (t_0, \cdots, t_n) \in \mathbb{R}^{n+1} \mid \cdots \}$$

$$\sigma|_{[v_0, \cdots \hat{v}_i \cdots v_n]}(t_0, \cdots, t_{n-1})$$

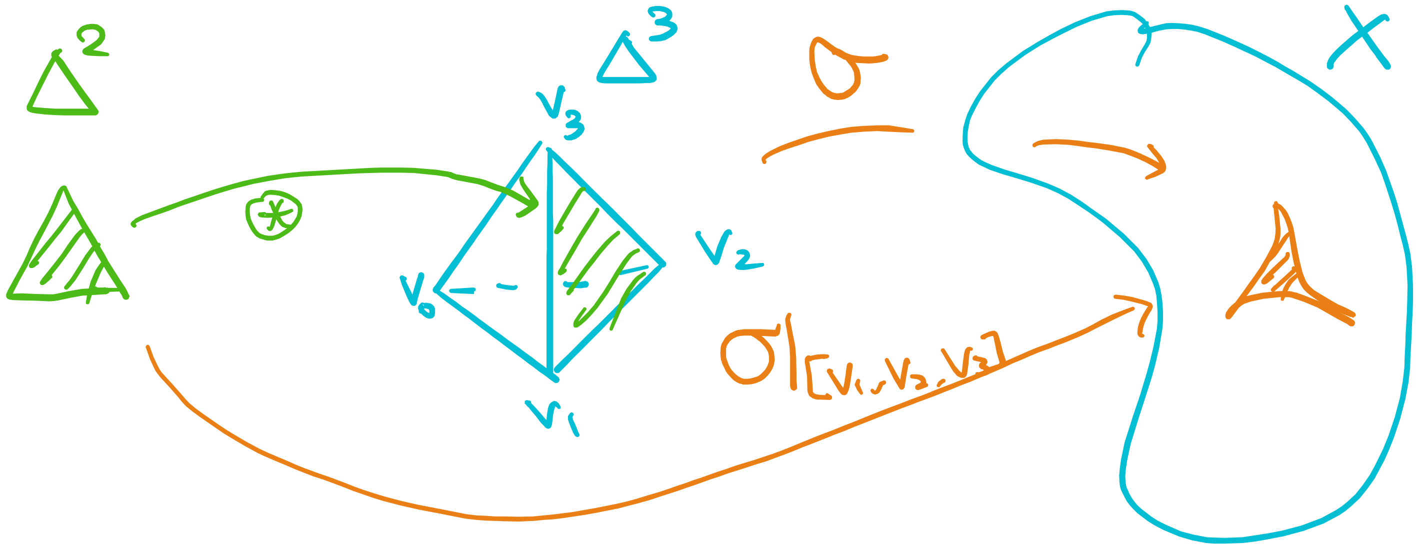
$$= \sigma\big(t_0 v_0 + t_1 v_1 + \cdots + t_{i-1} v_{i-1}$$

$v_0 = (1, 0, \cdots 0)$

$v_1 = (0,1,0,\cdots)$ $\qquad + t_i v_{i+1} + \cdots + t_{n-1} v_n \,)$

$\vdots$

$$= \sigma(t_0, t_1, \cdots, t_{i-1}, 0, t_i, \cdots, t_{n-1})$$



So we have

$$\cdots \longrightarrow C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \xrightarrow{\partial_{n-1}} C_{n-2}(X)$$

$$\rightarrow \cdots \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \rightarrow 0$$

which is called the singular chain complex of X.

Lemma

$$\partial_n \circ \partial_{n+1} = 0$$

$\forall\, n \geq 0.$

pf: Same computation (simplicial homology)

Def (p.108)

The n-th singular homology group of a space X is the quotient group

$$H_n(X) = \ker(\partial_n) / \operatorname{im}(\partial_{n+1})$$

An element in $\ker(\partial_n)$ is called a (singular) n-cycle.

An element in $\operatorname{im}(\partial_{n+1})$ is called a (singular) n-boundary.

Prop (Prop 2.8)

If X is a point, then

$$H_n(X) \cong \begin{cases} \mathbb{Z} & \text{if } n=0 \\ 0 & \text{if } n \geq 1. \end{cases}$$

pf

There is only one continuous map

$$\Delta^n \xrightarrow{*_n} X$$

for each n.

$$\Rightarrow \quad C_n(X) = \mathbb{Z} \cdot *_n \cong \mathbb{Z}$$

Furthermore,

$$*_n \big|_{[v_0, \cdots \hat{v_i} \cdots v_n]} = *_{n-1}$$

$$\Delta^{n-1} \to X$$

$*_{n-1}$

So

$$\partial_n(*_n) = \sum_{i=0}^{n} (-1)^i \, *_n \big|_{[v_0 \cdots \hat{v}_i \cdots v_n]}$$

$$= \underbrace{*_{n-1} - *_{n-1} + *_{n-1} - \cdots}_{n+1 \text{ terms}}$$

$$= \begin{cases} 0 & \text{if } n \text{ is odd} \\ *_{n-1} & \text{if } n \text{ is even} \end{cases}$$

Thus, the singular complex of X is isomorphic to

$$\cdots \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \cdots$$

$$\cdots \underset{\partial_2}{\xrightarrow{\cong}} \underset{n=1}{\mathbb{Z}} \underset{\partial_1}{\xrightarrow{0}} \underset{n=0}{\mathbb{Z}} \underset{\partial_0}{\longrightarrow} 0$$

$$\Rightarrow H_0(X) = \ker(\partial_0) \big/ \operatorname{im}(\partial_1) \cong \mathbb{Z}/0 \cong \mathbb{Z}$$

$$H_1(X) = \ker(\partial_1) \big/ \operatorname{im}(\partial_2) \cong \mathbb{Z}/\mathbb{Z} \cong 0$$

$$H_1(X) = \quad \overset{0}{\cong}\;/\,\mathrm{im}(\partial_2) \quad /\mathbb{Z} \quad 0$$

$$H_2(X) = \ker(\partial_2)\,/\,\mathrm{im}(\partial_3) \cong 0/0 \cong 0$$

So
$$H_n(X) \cong \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n \geq 1 \end{cases} \qquad \#$$

<u>Prop</u> (Prop 2.6)

If X_α are the path-connected components of a space X, then

$$H_n(X) \cong \bigoplus_\alpha H_n(X_\alpha)$$

pf: exercise.

sketch:

$$\begin{tikzcd}
\cdots \arrow[r] & C_n(X) \arrow[r, "\partial"] \arrow[d, phantom, "\cong"] & C_{n-1}(X) \arrow[r] \arrow[d, phantom, "\cong"] & \cdots \\
\arrow[r] & \bigoplus_\alpha C_n(X_\alpha) \arrow[r, "\bigoplus \partial_n^\alpha"] & \bigoplus_\alpha C_{n-1}(X_\alpha) \arrow[r] & \cdots
\end{tikzcd}$$

Prop (Prop 2.7)

If X is nonempty and path-connected, then

$$H_0(X) \cong \mathbb{Z}$$

If X is nonempty and X_α are the path-connected components of X, then

$$H_0(X) = \bigoplus_\alpha \mathbb{Z}$$

Remark

$C_0(X)$ = free abelian group generated by $\{\text{one point}\} = \Delta^0 \to X$

= free abelian group generated by X.

$\{\sum_{i=1}^{k} m_i x_i \mid x_i \in X, m_i \in \mathbb{Z}\}$

$$= \{ \sum_{i=1} m_i \cdot x_i \mid \ldots \}$$

Define

$$\varepsilon : C_0(X) \longrightarrow \mathbb{Z}$$

by
$$\varepsilon\Big(\sum_{i=1}^{k} m_i \cdot x_i\Big) = \sum_{i=1}^{k} m_i \in \mathbb{Z}$$

This is a group homomorphism

Furthermore, the composition

$$C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\varepsilon} \mathbb{Z}$$

is zero, because

$\sigma : \Delta^1 \to X$

$\Delta^0 \to X$, $* \mapsto \sigma(v_1)$

$\sigma(v_0)$

$$\varepsilon(\partial_1(\sigma)) = \varepsilon\Big(\underline{\underline{1}} \cdot \boxed{\sigma|_{[v_1]}} \underline{\underline{-}} \sigma|_{[v_0]}\Big)$$

$\cong \sigma(v_1)$ a point in X

$$= 1 - 1 = 0$$

Therefore we have

$$\cdots \to C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \to \cdots$$

$$\cdots \to C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

We can define another version of homology by this Complex

<u>Def</u> (p. 110)

The n-th reduced homology group $\tilde{H}_n(X)$ of a space X is

$$\tilde{H}_n(X) = \begin{cases} H_n(X) & \text{if } n \geq 1 \\ \ker(\varepsilon) / \operatorname{im}(\partial_1) & \text{if } n = 0 \end{cases}$$