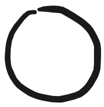
Algebraic Topology 2/27

Main object to study:

topological spaces ← set + open subsets

e.g. , , $\mathbb{R}^n$, subsets of $\mathbb{R}^n$,

metric space

Basic idea:

difficult things $\xrightarrow{\text{some construction}}$ easier things

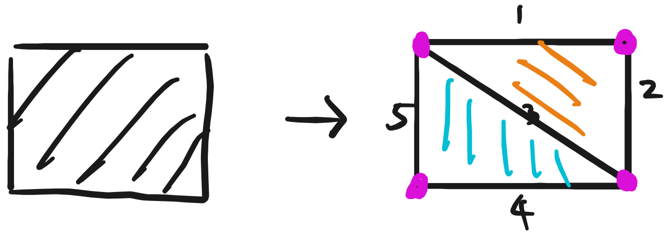
e.g. (topological) spaces — e.g. homology, cohomology — e.g. numbers, groups, vector sp

(typically) category — functor — category

As a concrete example of such construction, let us consider "Euler characteristic"

$$\chi = \underbrace{\#(\text{vertexes})}_{v} - \underbrace{\#(\text{edges})}_{e} + \underbrace{\#(\text{faces})}_{f}$$

e.g.

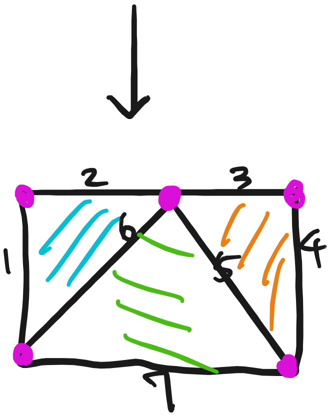


$v = 4$

$e = 5$

$f = 2$

$\Rightarrow \chi = 4 - 5 + 2 = 1$



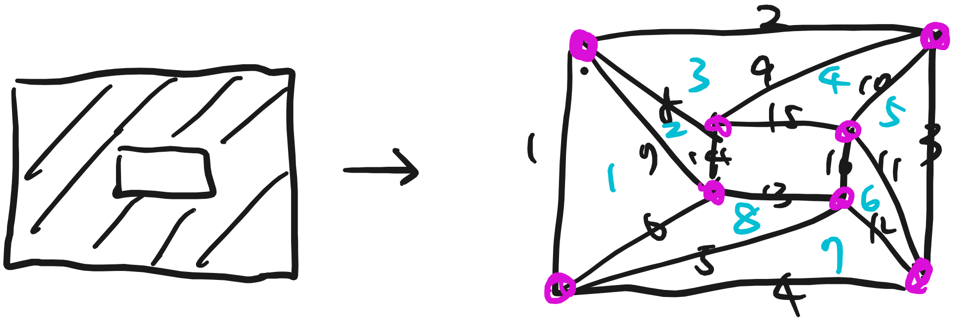
$v = 5$

$e = 7$

$f = 3$

$\Rightarrow \chi = 5 - 7 + 3 = 1$

e.g.



$v = 8$

$e = 16$

$f = 8$

$\Rightarrow \chi = 8 - 16 + 8 = 0$

Main goal of this course:

introduce a more complete theory behind Euler characterist

— the theory of (co)homology

Remark

$\exists$ many different (co)homology

e.g.

- simplicial (co)homology
- singular (co)homology
- Čech cohomology, sheaf cohomology
- de Rham cohomology
- K-theory
- Hochschild (co)homology (coh. of alg.)
- Lie algebra cohomology

and more

Simplicial homology

Here we continue to develop the idea of triangular decomposition.

Consider n-dimensional analogue

of triangle : n-simplex

Def (p.103)

Let $v_0, \cdots, v_n$ be points in $\mathbb{R}^n$ s.t.

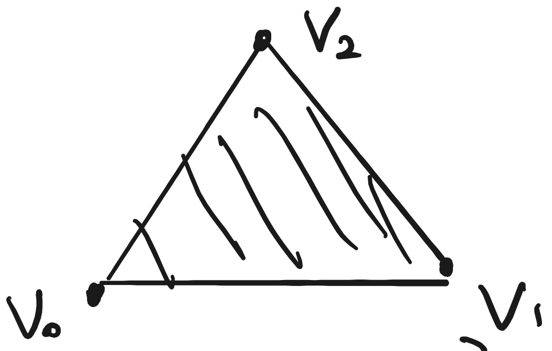
$$v_1 - v_0,\ v_2 - v_0,\ \cdots,\ v_n - v_0$$

are linearly independent. $\mathbb{R}^m$

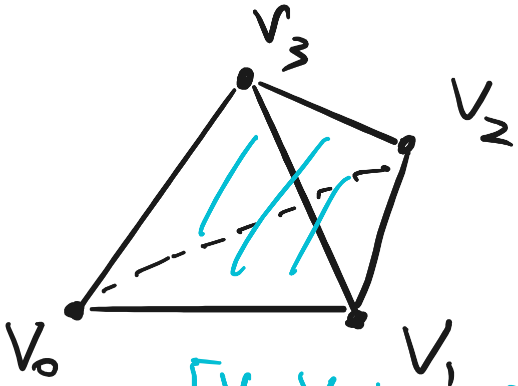
The n-simplex spanned by $v_0, \cdots, v_n$ is the ordered $(n+1)$-tuple

$$[v_0, v_1, \cdots, v_n]$$

Picture:

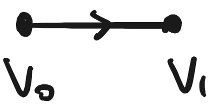


$[v_0, v_1, v_2]$

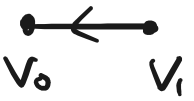


$[v_0, v_1, v_2, v_3]$

$[v_0, v_1]$



$[v_1, v_0]$



The standard n-simplex is

$$\Delta^n = [V_1, V_2, \cdots, V_{n+1}]$$

where

$$\underset{\overset{\parallel}{(1,0,\ldots,0)}}{V_1}, \cdots, \underset{\overset{\parallel}{(0,\cdots,0,1)}}{V_{n+1}} \in \mathbb{R}^{n+1}$$

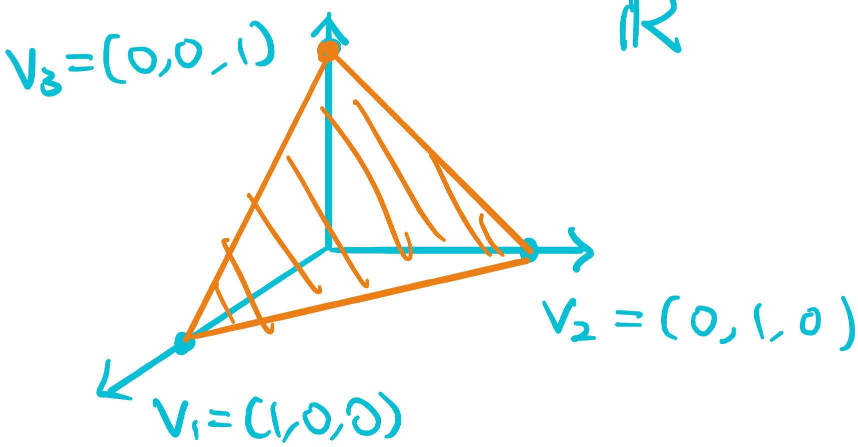
$V_i = (0\ldots, \overset{i\text{-th}}{1}, 0\ldots 0)$

is the standard basis

$$\Delta^n = \left\{ \overset{t_0,\cdots,t_n \in \mathbb{R}}{(t_0, t_1, \cdots, t_n)} \in \mathbb{R}^{n+1} \;\middle|\; \begin{array}{l} \sum_{i=0}^{n} t_i = 1 \\ t_i \geq 0 \end{array} \right\}$$

$\mathbb{R}^3$

Take $m = n+1$



An n-simplex $[V_0, \cdots, V_n]$ is often considered as the set

$$[V_0, \cdots, V_n] = \left\{ \overset{t_i \in \mathbb{R}}{t_0 V_0 + t_1 V_1 + \cdots + t_n V_n} \in \mathbb{R}^m \;\middle|\; \begin{array}{l} \sum_i t_i = 1 \\ t_i \geq 0 \end{array} \right\}$$

The coefficients t_i are called the barycentric coordinates of the point

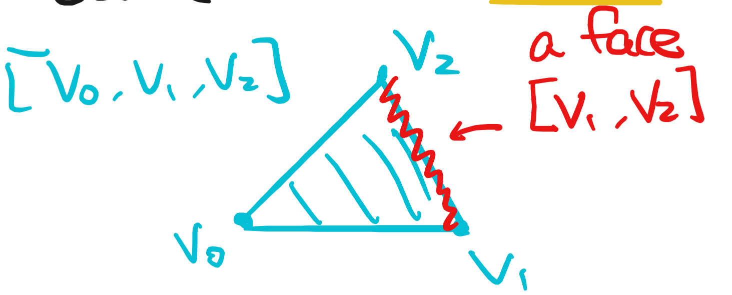
$\sum_i t_i v_i$ in $[v_0, \cdots, v_n]$

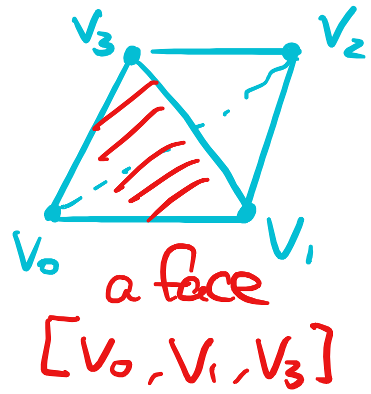
The $n+1$ points $v_0, \cdots, v_n$ are called the **vertexes** of $[v_0, \cdots, v_n]$.

If we delete v_i one of $v_0, \cdots, v_n$, then the remaining vertexes span an $(n-1)$-simplex, denoted by

$$[v_0, \cdots \hat{v_i} \cdots, v_n]$$

called a **face** of $[v_0, \cdots, v_n]$.





The union of all the faces of Δ^n is called the **boundary** of Δ^n, denoted by $\partial\Delta^n$.

The open simplex Δ^n is

$$\mathring{\Delta}^n = \Delta^n - \partial\Delta^n$$

Def (page 103)

Let X be a topological sp.

A Δ-complex structure on X

is a collection of maps

$$\sigma_\alpha : \Delta^n \longrightarrow X$$ (n depends on α)

s.t.

(i) The restriction

$$\sigma_\alpha|_{\mathring{\Delta}^n} : \mathring{\Delta}^n \longrightarrow X$$

is 1-1, and each point in X is in the image of exactly one such restriction

$\sigma_\alpha|\mathring{\Delta}^n$.

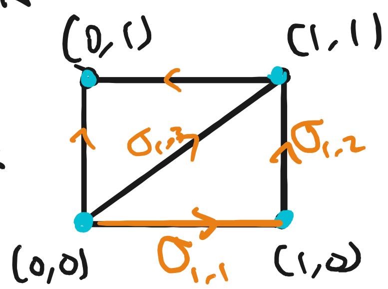
(ii) Each restriction of σ_α to a face of Δ^n is one of the maps $\sigma_\beta : \Delta^{n-1} \to X$.

(iii) A subset $A \subset X$ is open iff $\sigma_\alpha^{-1}(A)$ is open in Δ^n for each α.

<u>Example</u>

$\mathbb{R}^2$

$X =$



$\sigma_{0,1}, \sigma_{0,2}, \sigma_{0,3}, \sigma_{0,4}$:

$\Delta^0 = \{*\} \longrightarrow X$,

$\sigma_{0,1}(*) = (0,0)$,

$\sigma_{0,2}(*) = (1,0)$,

$\sigma_{0,3}(*) = (1,1)$

$\sigma_{0,4}(*) = (0,1)$

$\sigma_{1,1}, \sigma_{1,2}, \sigma_{1,3}, \sigma_{1,4}, \sigma_{1,5} : \underbrace{\Delta^1}_{\{(t_0,t_1)\in\mathbb{R}^2 \mid t_0+t_1=1,\ t_0,t_1\geq 0\}} \longrightarrow X.$

$\sigma_{1,1}(t_0,t_1) = (t_1, 0)$, $\sigma_{1,2}(t_0,t_1) = (t_0+t_1, t_1)$,

$\sigma_{1,3}(t_0,t_1) = (t_1,t_1)$, $\sigma_{1,4}(t_0,t_1) = (t_0, t_0+t_1)$,

$\sigma_{1,5}(t_0,t_1) = (0,t_1)$

$\sigma_{2,1}, \sigma_{2,2} : \Delta^2 = \{(t_0,t_1,t_2) \in \mathbb{R}^3 \mid t_0+t_1+t_2 = 1\} \to X$

$\sigma_{2,1}, \sigma_{2,2} : \Delta = \{(t_0, t_1, t_2) \in \mathbb{R} \mid \ldots \; t_0, t_1, t_2 \geq 0\} \to \ldots$

$\sigma_{2,1}(t_0, t_1, t_2) = (t_1 + t_2, t_2)$

$\sigma_{2,2}(t_0, t_1, t_2) = (t_1, t_1 + t_2)$

<u>Recall</u>

The topology of a space generated by a "gluing process" is described by the quotient space topology

Let X be a space, "$\sim$" an equivalence relation on X, and $\pi : X \to X/\sim$ the natural projection.

"A subset $U \subset X/\sim$ is open iff $\pi^{-1}(U)$ is open in X."

This defines a topology on $X/\sim$.

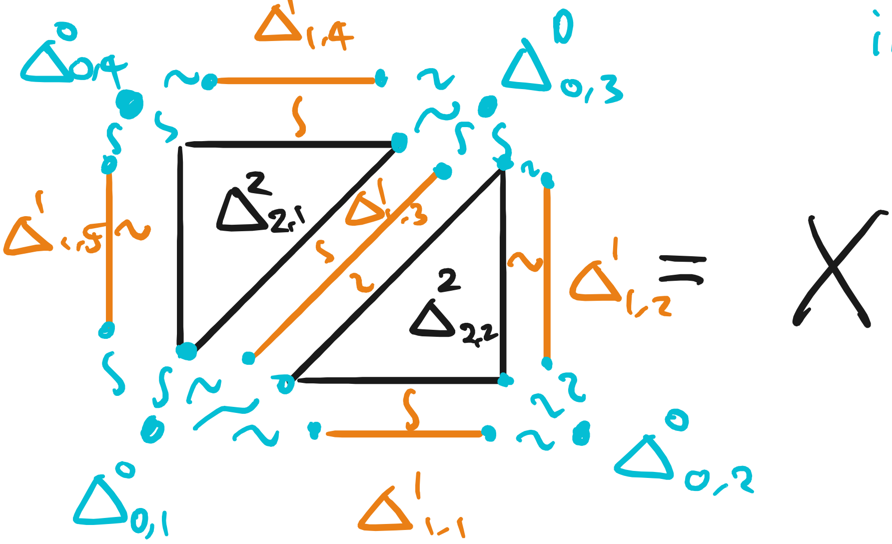
This topology is called the quotient space topology of $X/\sim$

Condition (iii) $\Rightarrow$

$$X = \coprod_{\alpha} \Delta_{\alpha}^{n_\alpha} / \sim$$

$x \sim y \iff \sigma_\alpha(x) = \sigma_\beta(y)$ in X

e.g.



Simplicial homology

Let X be a Δ-complex. Define

free abelian group

$\Delta_n(X) =$ with basis $\sigma_\alpha : \Delta^n \to X$

Elements in $\Delta_n(X)$, called (simplicial) n-chains, can be written as finite formal sums

$\sigma_\alpha : \Delta^n \to X$

$$\sum_\alpha m_\alpha \cdot \sigma_\alpha \qquad m_\alpha \in \mathbb{Z}$$

which are subject to the rules:

(i) $m_\alpha = 0$ except finite α

(ii) $\sum_\alpha m_\alpha \cdot \sigma_\alpha = \sum_\alpha m'_\alpha \sigma_\alpha$

$\iff m_\alpha = m'_\alpha \quad \forall \alpha$

(iii) $\sum_\alpha m_\alpha \cdot \sigma_\alpha + \sum_\alpha m'_\alpha \cdot \sigma_\alpha$

$= \sum_\alpha (m_\alpha + m'_\alpha) \cdot \sigma_\alpha$

Def (p.105)

The boundary homomorphism

The boundary homomorphism

$$\partial_n : \Delta_n(X) \longrightarrow \Delta_{n-1}(X)$$

is the group homomorphism with the property

$$\partial_n(\sigma_\alpha) = \sum_{i=0}^{n} (-1)^i \, \sigma_\alpha \big|_{[v_0, \dots \hat{v_i} \dots, v_n]}$$

We have the diagram

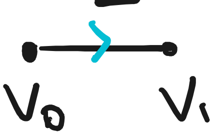
$$\cdots \to \Delta_n(X) \xrightarrow{\partial_n} \Delta_{n-1}(X) \xrightarrow{\partial_{n-1}} \Delta_{n-2}(X) \to \cdots$$

$$\cdots \to \Delta_1(X) \xrightarrow{\partial_1} \Delta_0(X) \to 0$$

called the simplicial chain complex of X

Picture:

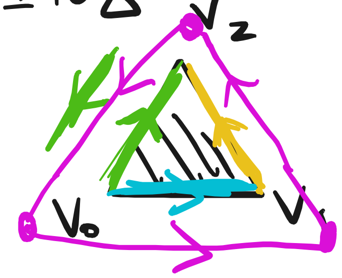
① $\sigma = id_{\Delta^1}$ $\xrightarrow{\partial_1}$ $id\big|_{\underbrace{[\hat{v_0}, v_1]}_{[v_1]}} - id\big|_{\underbrace{[v_0, \hat{v_1}]}_{[v_0]}}$



$\Delta^1 = [v_0, v_1] = X$

" $v_1 - v_0$ "

② $\sigma = id_{\Delta^2}$



"boundary"

$\xrightarrow{\partial_2}$ $\underbrace{id|_{[\hat{v}_0, v_1, v_2]}}_{[v_1, v_2]} - \underbrace{id|_{[v_0, \hat{v}_1, v_2]}}_{[v_0, v_2]} + \underbrace{id|_{[v_0, v_1, \hat{v}_2]}}_{[v_0, v_1]}$

" $[v_1, v_2] - [v_0, v_2] + [v_0, v_1]$ "

Lemma (Lemma 2.1)

The composition " $\partial \circ \partial = 0$ "

$$\Delta_n(X) \xrightarrow{\partial_n} \Delta_{n-1}(X) \xrightarrow{\partial_{n-1}} \Delta_{n-2}(X)$$

is zero: $\partial_{n-1} \circ \partial_n = 0 \quad \forall n$

$\Rightarrow \ker(\partial_{n-1}) \supseteq \operatorname{im}(\partial_n)$

abelian group

Pf

$$(\partial_{n-1} \circ \partial_n)(\sigma)$$

$$= \partial_{n-1}\left(\sum_{i=1}^{n} (-1)^i \, \sigma|_{[v_0, \cdots \hat{v}_i \cdots v_n]} \right)$$

n

$$= \sum_{i=1} \underline{(-1)^i}\ \underbrace{\partial_{n-1}\left(\sigma|_{[v_0 \cdots \hat{v}_i \cdots v_n]}\right)}_{=}$$

$$\sum_{j=0}^{i-1} \underline{(-1)^j} \cdot \sigma|_{[v_0, \cdots \hat{v}_j \cdots \hat{v}_i \cdots v_n]}$$

$$+ \sum_{j=i+1}^{n} \underline{\underline{(-1)^{j-1}}} \cdot \sigma|_{[v_0 \cdots \hat{v}_i \cdots \hat{v}_j \cdots v_n]}$$

$$= \sum_{j<i} (-1)^{i+j}\ \sigma|_{[v_0 \cdots \hat{v}_j \cdots \hat{v}_i \cdots v_n]}$$

$$+ \sum_{\substack{j>i \\ = \ = \\ i \ \ j}} (-1)^{i+j-1}\ \sigma|_{[v_0 \cdots \hat{v}_{\underset{j}{i}} \cdots \hat{v}_{\underset{i}{j}} \cdots v_n]}$$

$$= 0 \qquad \#$$

<u>Def</u> (p. 106)

Let X be a Δ-complex.

The quotient group

$$H_n^{\Delta}(X) = \ker(\partial_n) \big/ \operatorname{im}(\partial_{n+1})$$

is called the n-th simplicial homology group of X.

Recall:

Every finitely generated abelian group is iso. to a group of the form

$$\mathbb{Z}^{r} \oplus \mathbb{Z}_{q_1} \oplus \cdots \oplus \mathbb{Z}_{q_i},$$

and the number r is called the rank of this abelian group.

Def

Assume X is a Δ-complex with finite σ_α. The n-th Betti number of X is $\operatorname{rank} H_n^{\Delta}(X)$

Betti number of X is rank $H_n^\Delta(X)$.

The Euler characteristic of X is

$$\chi(X) = \sum_{n=0}^{\infty} (-1)^n \operatorname{rank} H_n^\Delta(X)$$

Remark

$H_n^\Delta(X)$ only depends on the topology of X, independent of the choice of Δ-complex structure on X.

We skip the proof here. Instead, we will prove more general theorems from the point of view of "singular homology."

Good: straightforward for computation

Bad: difficult for developing a general

theory.

Example

① $X = \{x_0\}$

$$\sigma : \begin{array}{ccc} \Delta^0 & \longrightarrow & X \\ \Psi & & \Psi \\ * & \longmapsto & x_0 \end{array}$$

$$0 \to 0 \to \Delta_1(X) = \{0\} \xrightarrow{\partial_1} \Delta_0(X) = \mathbb{Z}\sigma \cong \mathbb{Z} \xrightarrow{\partial_0} 0$$

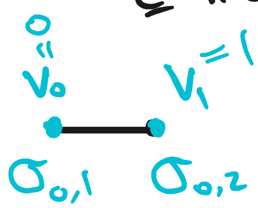
$\ker(\partial_0) = \mathbb{Z}\cdot\sigma$

$\mathrm{im}(\partial_1) = 0$

$$\ker(\partial)/\mathrm{im}(\partial_1)$$

$$\Rightarrow H_n^\Delta(X) \cong \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n > 0 \end{cases}$$

② $X = [0,1] \subseteq \mathbb{R}$



$\sigma_{0,1}, \sigma_{0,2} : \Delta^0 \longrightarrow X$

$\sigma_{0,1}(*) = v_0, \quad \sigma_{0,2}(*) = v_1$

$\sigma_1 : \Delta^1 \longrightarrow X$

$\sigma_1(t_0, t_1) = t_1 \in X \subseteq \mathbb{R}$

$t_0 + t_1 = 1 \Rightarrow 0 \leq t_1 \leq 1$

$t_0, t_1 \geq 0$

$\partial_2 \qquad \partial_1 \qquad \partial_0$

$$\cdots 0 \to \Delta_2(X) \to \Delta_1(X) = \mathbb{Z}\sigma_1 \to \Delta_0(X) \to 0$$

$$\Delta_2(X) = 0, \qquad \Delta_0(X) = \mathbb{Z}\sigma_{0,1} \oplus \mathbb{Z}\sigma_{0,1}$$

$$\varphi$$

$$\sigma_1 \longmapsto \sigma_{0,2} - \sigma_{0,1}$$

or $(-1, 1)$

$\cong$

$$0 \to \mathbb{Z} \xrightarrow{\partial_1} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\partial_0} 0$$

$$1 \longmapsto (-1, 1)$$

$$-a \longmapsto \overline{(-a, a)} = (0,0)$$

$$H_0^{\Delta}(X) = \ker \partial_0 / \operatorname{im} \partial_1 = \mathbb{Z} \oplus \mathbb{Z} / \langle (-1, 1) \rangle \cong \mathbb{Z}$$

$$H_1^{\Delta}(X) = \ker \partial_1 / \underbrace{\operatorname{im} \partial_2}_{=0} = 0, \qquad \ker \partial_1 = 0$$

So

$$H_n^{\Delta}(X) \cong \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n > 0 \end{cases}$$

\#