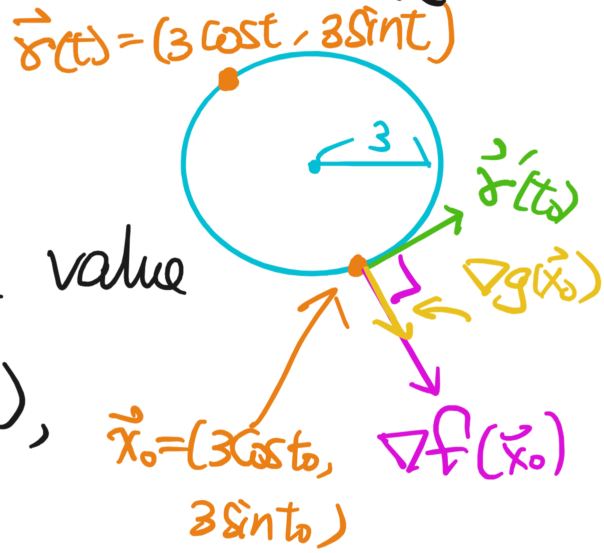
Calculus 5/28

Lagrange multiplier

Last time, we wanted to find the extreme values of f on the circle

$g(x,y) = x^2 + y^2 = 9.$



If f has an extreme value at $\vec{x}_0 = (3\cos t_0, 3\sin t_0)$,

then

$$\left.\frac{d}{dt}\right|_{t_0} f(\vec{\gamma}(t)) = 0$$

$$\overset{\text{chain}}{\underset{\text{rule}}{=}} \nabla f(\vec{x}_0) \bullet \vec{\gamma}'(t_0) = 0$$

Therefore, to find the extreme values of $\boxed{f(x,y)}$ on $x^2+y^2 = g(x,y) = 3^2$, it suffices

$= x^2+y^2-2x-2y+4$

to check the points at which

∇f and ∇g

∇f and ∇g

are parallel.

<u>Example</u>

Find the extreme values of

$$f(x,y) = x^2+y^2-2x-2y+4$$

on the circle

$$g(x,y) = x^2+y^2 = 3^2$$

<u>sol</u>

Solve (x_0, y_0) with the property

$$\nabla f(x_0,y_0) = (2x_0-2,\ 2y_0-2)$$

is parallel to

$$\nabla g(x_0,y_0) = (2x_0,\ 2y_0)$$

That is,

$$(2x_0-2)\,2y_0 = (2y_0-2)\,2x_0$$

$$\shortparallel \qquad\qquad\qquad \shortparallel$$

$$\cancel{4x_0y_0}-4y_0 \qquad \cancel{4x_0y_0}-4x_0$$

$\Leftrightarrow \quad x_0 = y_0$

Since (x_0, y_0) is on the circle

$$x^2+y^2=3^2,$$

we have

$$(x_0, y_0) = \left(\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right) \text{ or } \left(-\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right).$$

Then check

$$f\left(\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right) \text{ and } f\left(-\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right) \quad \#$$

<u>Remark</u>

If $\vec{x}_0$ maximizes (or minimizes) $f(\vec{x})$ subject to the side condition

$$g(\vec{x}) = 0,$$

then

$$\nabla f(\vec{x}_0) \text{ and } \nabla g(\vec{x}_0)$$

are parallel.

Then, if $\nabla g(\vec{x}_0) \neq \vec{0}$, then $\exists\, \lambda$ s.t.

$$\nabla f(\vec{x}_0) = \lambda \cdot \nabla g(\vec{x}_0)$$

Such a scalar λ is called a Lagrange multiplier.

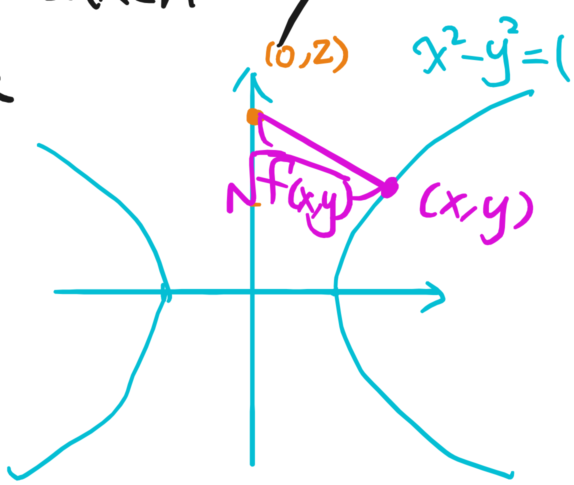
Example

Find the minimum value taken by

$$f(x,y) = x^2 + (y-2)^2$$

on

$$x^2 - y^2 = 1$$



sol

Set

$$g(x,y) = x^2 - y^2 - 1$$

Minimize

$$f(x,y) = x^2 + (y-2)^2 \quad \text{subject to } g(x,y)=0$$

Solve (x,y) s.t.

$$\nabla f(x,y) = (2x,\ 2(y-2))$$

is prartlel to

$$\nabla g(x,y) = (2x, -2y)$$

That is,

$$-2x \cdot 2y = 2(y-2)\cdot 2x = 4xy - 8x$$

$$8xy - 8x = 0 \;\Rightarrow\; x = 0 \text{ or } y = 1$$

$$\| $$

$$8x(y-1)$$

① $\begin{cases} x = 0 \\ x^2 - y^2 = 1 \end{cases} \Rightarrow -y^2 = 1 \quad \rightarrow\leftarrow$ No such point!

② $\begin{cases} y = 1 \\ x^2 - y^2 = 1 \end{cases} \Rightarrow \begin{cases} x = \pm\sqrt{2} \\ y = 1 \end{cases} \Rightarrow (\sqrt{2}, 1)$ and $(-\sqrt{2}, 1)$

$$f(\sqrt{2}, 1) = (\sqrt{2})^2 + (1-2)^2 = 3$$

$$f(-\sqrt{2}, 1) = (-\sqrt{2})^2 + (1-2)^2 = 3$$

by picture ↙

So

$3 = f(\pm\sqrt{2}, 1)$ is the minimum value #

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§ Integration of functions of several variables

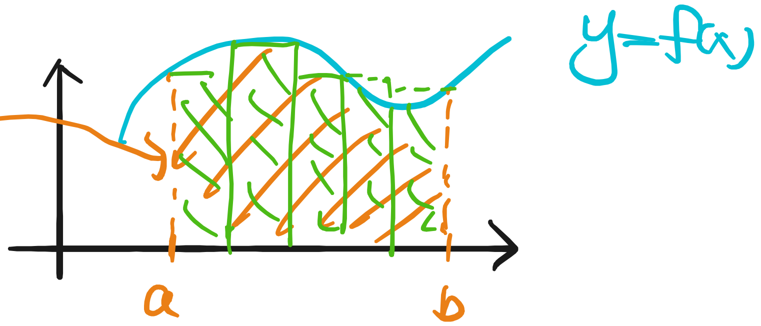
Double integral

Recall:

For a positive function $f(x)$, the integral

$$\int_a^b f(x)\,dx$$

is the area of the region

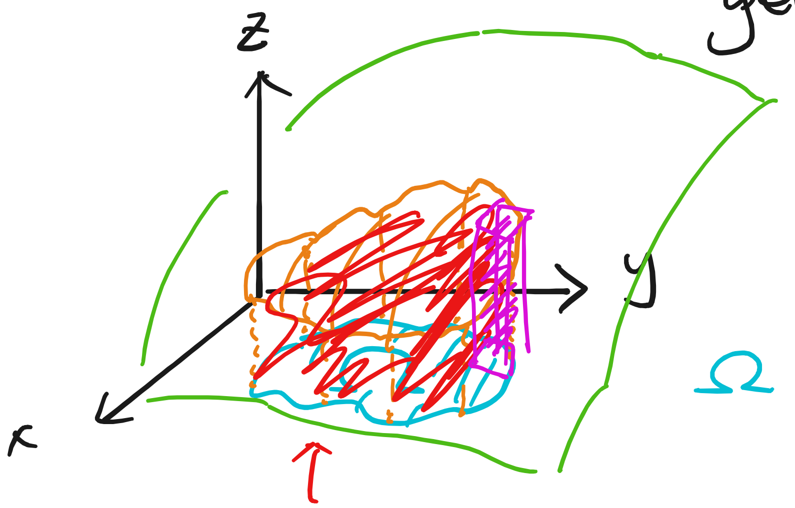


Parallelly, the integral

$$\iint f(x,y)\,dxdy$$

$\iint_{\Omega}$

is the volume of the solid bounded by $z = f(x,y)$, Ω and the cylinder generated by Ω



$z = f(x,y)$

Ω is in xy-plane

volume $= \iint_{\Omega} f(x,y)\, dxdy$

In particular,

$$\iint_{\Omega} 1 \, dxdy = \iint_{\Omega} dxdy$$

is the area of Ω.

Such a double integral can be

such a double integral can be defined by a limit of "2-dimensional" Riemann sums.

Too complicated, we skip it here.

in Marsden's book "Elementary classical analysis"

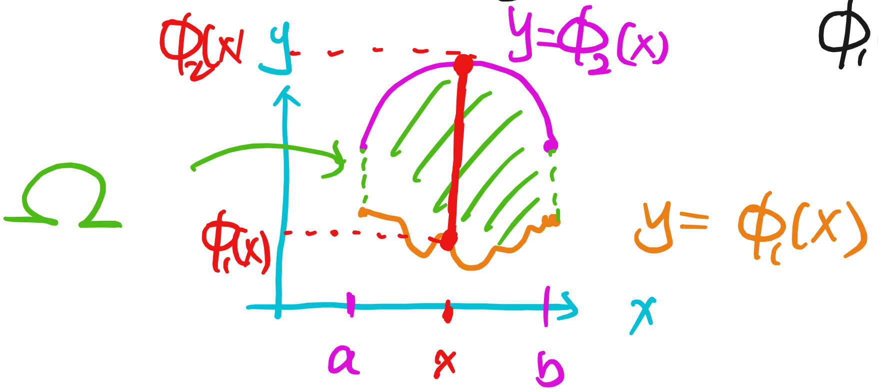
Thm (Cor 9.2.2)

Let $\phi_1, \phi_2 : [a,b] \to \mathbb{R}$ be continuous functions s.t.

$$\phi_1(x) \leq \phi_2(2) \qquad \forall x \in [a,b]$$

Let

$$\Omega = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \begin{array}{l} a \leq x \leq b, \\ \phi_1(x) \leq y \leq \phi_2(x) \end{array} \right\}$$



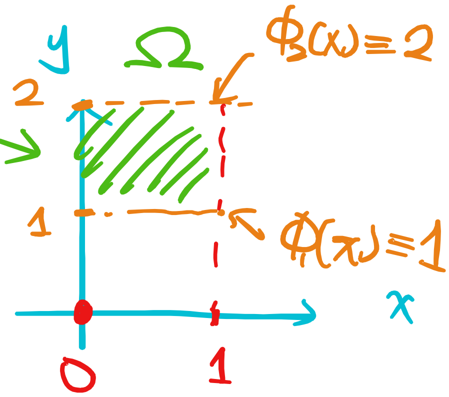
If $f : \Omega \to \mathbb{R}$ is continuous, then

$$\iint_\Omega f(x,y)\, dx dy = \int_a^b \left(\int^{\phi_2(x)} f(x,y)\, dy \right) dx$$

$$\iint_\Omega f(x,y)\,dx\,dy = \int_a \left(\int_{\phi_1(x)} f(x,y)\,dy \right) dx$$

Example

$$\Omega = [0,1] \times [1,2]$$



$$\iint_\Omega xy\,dx\,dy = \int_0^1 \left(\int_1^2 xy\,dy \right) dx$$

$$= \int_0^1 x\frac{y^2}{2}\Big|_{y=1}^{2} = x\left(\frac{2^2}{2} - \frac{1^2}{2}\right) dx$$

$$= \int_0^1 \frac{3}{2}x\,dx = \frac{3}{2}\frac{x^2}{2}\Big|_{x=0}^{1}$$

$$= \frac{3}{4}$$ #