# Calculus 5/21

Recall

If $g$ is a smooth function of one variable and $g'(x_0) = 0$, then

(i) $g''(x_0) > 0 \Rightarrow g$ has a local minimum at $x_0$

(ii) $g''(x_0) < 0 \Rightarrow g$ has a local maximum at $x_0$

Thm (Thm 16.5.3)

Suppose $f = f(x,y)$ has continuous second-order partial derivatives near $(x_0, y_0)$ and

$$\nabla f(x_0, y_0) = 0$$

Set

$$A = \frac{\partial^2 f}{\partial x^2}(x_0, y_0), \quad B = \frac{\partial^2 f}{\partial x \partial y}(x_0, y_0),$$

$$C = \frac{\partial^2 f}{\partial y^2}(x_0, y_0)$$

判别式

and form the discriminant

$$\boxed{D = AC - B^2.} = \det\begin{pmatrix} \underbrace{\frac{\partial^2 f}{\partial x^2}(x_0,y_0)}_{A} & \underbrace{\frac{\partial^2 f}{\partial x \partial y}(x_0,y_0)}_{B} \\ \underbrace{\frac{\partial^2 f}{\partial y \partial x}(x_0,y_0)}_{B} & \underbrace{\frac{\partial^2 f}{\partial y^2}(x_0,y_0)}_{C} \end{pmatrix}$$

(i) If $D<0$, then $(x_0, y_0)$ is a saddle point. ← " $\begin{pmatrix} A & B \\ B & C \end{pmatrix}$ is indefinite"

(ii) If $D>0$, then

(ii-1) $A > 0 \Rightarrow$ $f$ has a local minimum at $(x_0, y_0)$

↑ " $\begin{pmatrix} A & B \\ B & C \end{pmatrix}$ is positive definite " 正定

(ii-2) $A < 0 \Rightarrow$ $f$ has a local maximum at $(x_0, y_0)$

↑ " $\begin{pmatrix} A & B \\ B & C \end{pmatrix}$ is negative definite " 负定

Example

① $f(x,y) = x^2 + y^2$

Step 1: Find critical point(s)

$$Df = (2x, 2y) = (0,0)$$
$$\Leftrightarrow x=0, y=0$$

$(0,0)$ is the only critical point of $f$

Step2: Determine $f$ has a max, min or neither at $(0,0)$.

Apply Thm:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial(2x)}{\partial x} = 2 = A > 0$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial(2x)}{\partial y} = 0 = B$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial(2y)}{\partial y} = 2 = C$$

$$\Rightarrow D = AC - B^2 = 2 \cdot 2 - 0 = 4 > 0$$

$A > 0 \Rightarrow f$ has a local min. at $(0,0)$ #

(2) $f(x,y) = -x^2 - y^2$ $\quad (\nabla f = (-2x, -2y))$

$\Rightarrow$ critical point: $(0,0)$

$A = -2, \quad B = 0, \quad C = -2$

$D = (-2)(-2) - 0^2 = 4 > 0$

$A = -2 < 0 \Rightarrow f$ has a local max. at $(0,0)$ #

(3) $f(x,y) = xy$

Step1: critical point.

$\nabla f = (y, x) = (0,0)$

$\Leftrightarrow (x,y) = (0,0) \leftarrow$ critical point

Step2:

$A = f_{xx} = \frac{\partial y}{\partial x} = 0$

$B = f_{xy} = \frac{\partial y}{\partial y} = 1$

$C = f_{yy} = \frac{\partial x}{\partial y} = 0$

$$D = AC - B^2 = 0 \cdot 0 - 1^2 = -1 < 0$$

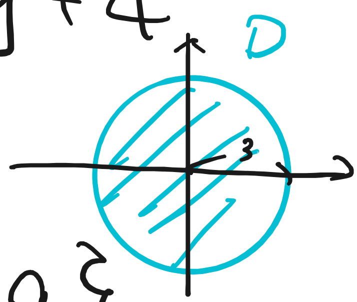
$\Rightarrow$ f has a saddle point at $(0,0)$. #

Example

Find the absolute extreme values taken by the function

$$f(x,y) = x^2 + y^2 - 2x - 2y + 4$$

on the closed disk




$$D = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \le 9\}$$

sol

Recall ( one-variable case )

$g = g(x)$. To find extreme value(s) of $g(x)$ on $[a,b]$

we consider

- values of $g$ at critical point(s)
- $g(a)$, $g(b)$

**Step 1**: Find the critical point(s) in the interior

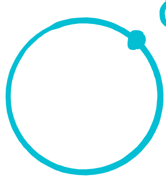
$$\{ x^2+y^2 < 9 \}$$

$$\nabla f = (2x-2, \ 2y-2) = (0,0)$$

$$\Leftrightarrow (x,y) = (1,1)$$ ← only critical point

**Step 2**: Find the extreme values on the boundary

$$x^2+y^2 = 9$$



$\vec{\gamma}(t) = (3\cos t, 3\sin t)$

The values of f on the boundary can be expressed by

$$F(t) = f(\vec{\gamma}(t)) = f(3\cos t, 3\sin t)$$
$$= 13 - 6\cos t - 6\sin t$$

$$F'(t) = 6\sin t - 6\cos t = 0$$

$$\Leftrightarrow t = \frac{\pi}{4} + n\pi.$$

Thus, the extreme values of f on the boundary may occur at

$$\vec{r}\left(\frac{\pi}{4}+n\pi\right) = \left(\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right), \left(-\frac{3\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2}\right)$$

<u>Step3</u> Compare the values of $f$ at all the candidate points:

$$f(1,1) = 1^2+1^2-2\cdot 1-2\cdot 1+4 = 2$$

$$f\left(\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right) = \frac{18}{4}+\frac{18}{4}-3\sqrt{2}-3\sqrt{2}+4 = 13-6\sqrt{2}$$
$$\approx 4.51$$

$$f\left(-\frac{3\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2}\right) = 13+6\sqrt{2} \approx 21.49$$

So the absolute maximum of $f$ is

$$13+6\sqrt{2} = f\left(-\frac{3\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2}\right)$$

and the absolute minimum of $f$ is

$$2 = f(1,1)$$

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HW8.8

$f = f(r)$ — one variable

$$g(x,y,z) = f\left(\sqrt{x^2+y^2+z^2}\right)$$

$\nabla g = ?$

$= (g_x, g_y, g_z)$

<u>Sol</u>

$\frac{1}{2}(x^2+y^2+z^2)^{-\frac{1}{2}} \cdot 2x$

$$g_x = \frac{\partial}{\partial x}\left(f\left(\sqrt{x^2+y^2+z^2}\right)\right)$$

$$\underset{\text{rule}}{\overset{\text{chain}}{=}} f'\left(\sqrt{x^2+y^2+z^2}\right) \cdot \frac{\partial (x^2+y^2+z^2)^{\frac{1}{2}}}{\partial x}$$

$$= f'\left(\sqrt{x^2+y^2+z^2}\right) \cdot \frac{x}{\sqrt{x^2+y^2+z^2}}$$

Similarly,

$$g_y = f'\left(\sqrt{x^2+y^2+z^2}\right) \cdot \frac{y}{\sqrt{x^2+y^2+z^2}}$$

$$g_z = f'\left(\sqrt{x^2+y^2+z^2}\right) \cdot \frac{z}{\sqrt{x^2+y^2+z^2}}$$

$$\Rightarrow \nabla g = (g_x, g_y, g_z)$$

$$= \left(f'(r) \cdot \frac{x}{r}, \ f'(r)\frac{y}{r}, \ f'(r)\frac{z}{r}\right)$$

$$= \frac{f'(r)}{r} \cdot (x, y, z) = f'(r)\frac{\vec{x}}{r}$$

#

HW4.1.

$f = f(x,y)$ is smooth

Prove

$$\frac{\partial^3 f}{\partial x^2 \partial y} = \frac{\partial^3 f}{\partial x \partial y \partial x} = \frac{\partial^3 f}{\partial y \partial x^2} = f_{xxy}$$

pf $\quad f_{yxx} \qquad f_{xyx}$

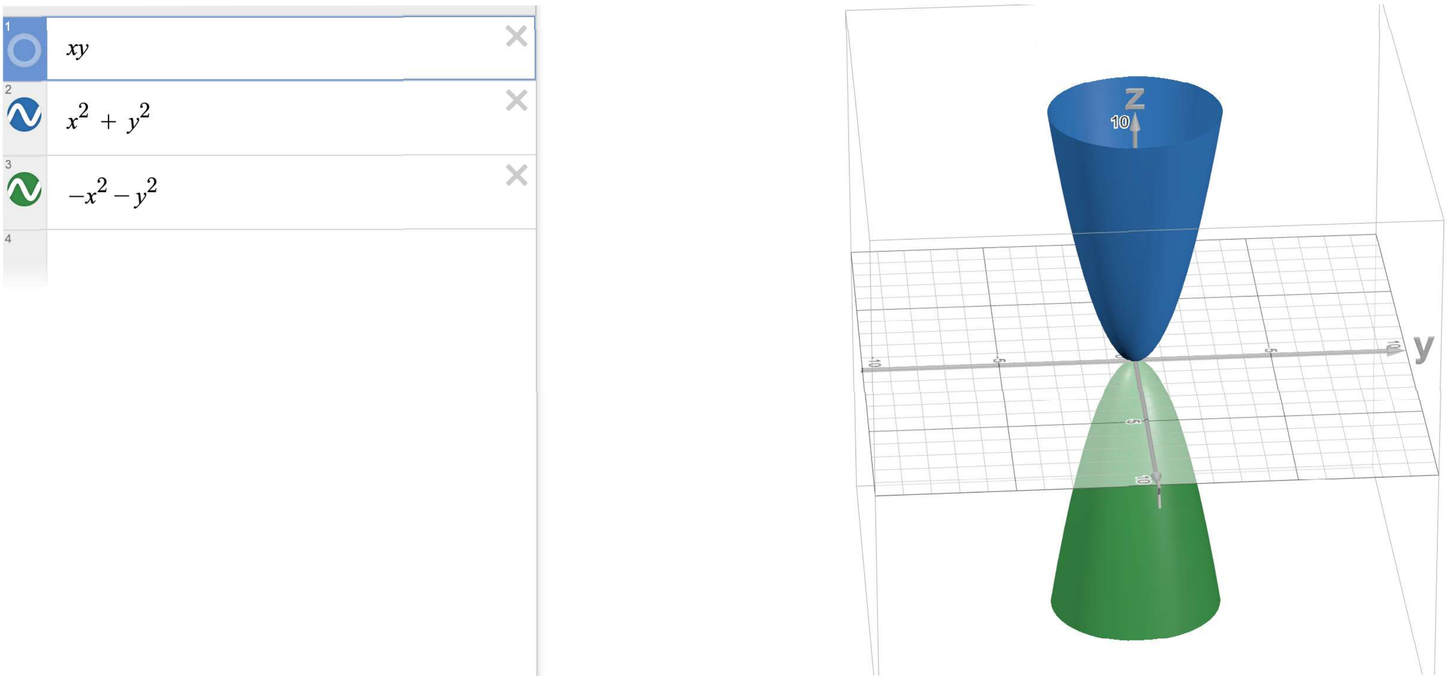
Recall if $g = g(x,y)$ is smooth, then

$$g_{xy} = g_{yx} \quad \leftarrow \text{our Thm.}$$

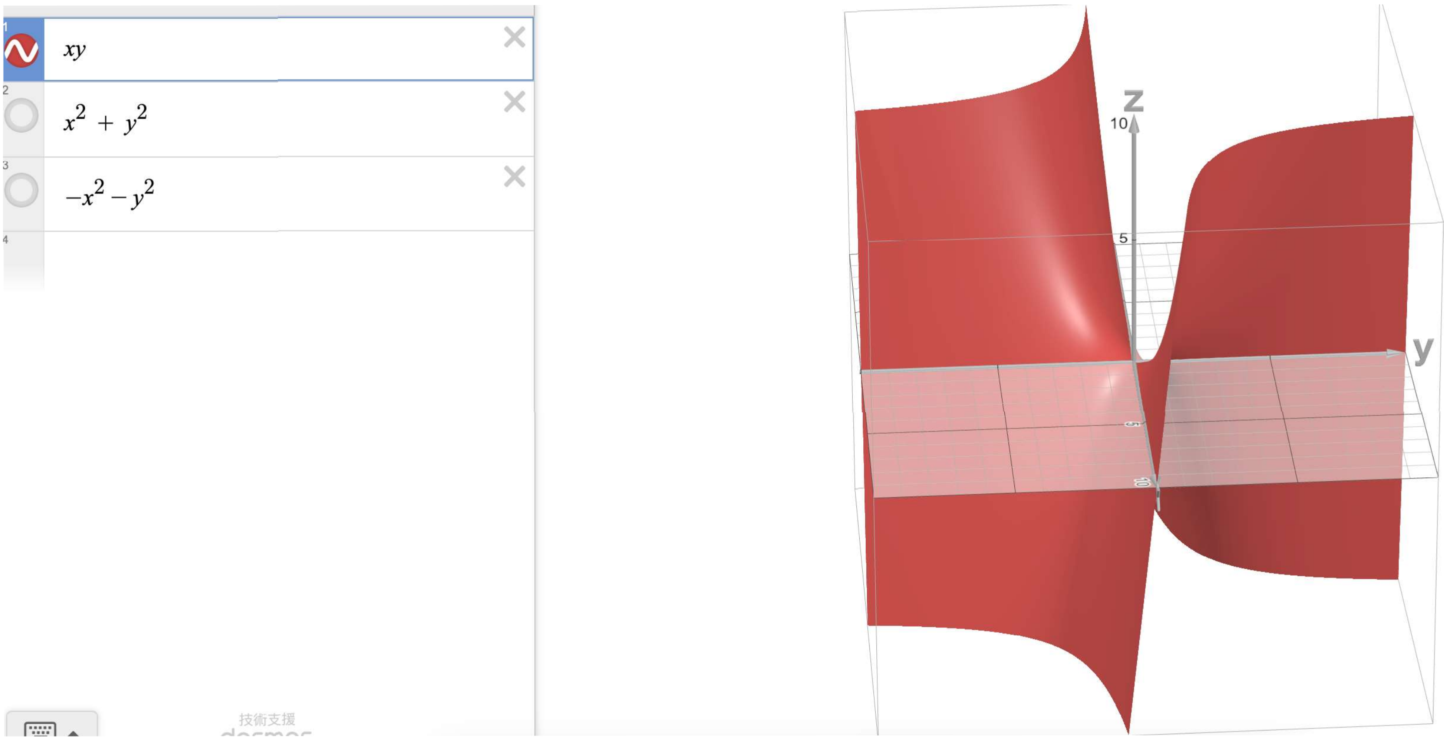
$$f_{xxy} = (f_x)_{xy} \overset{\text{Thm}}{=} (f_x)_{yx} = f_{xyx}$$

$$= (f_{xy})_x \overset{\text{Thm}}{=} (f_{yx})_x = f_{yxx}$$

#



1 xy
2 x^2 + y^2
3 −x^2 − y^2
4
z
10
y

1 xy
2 x^2 + y^2
3 −x^2 − y^2
4
z
10
5
y
技術支援