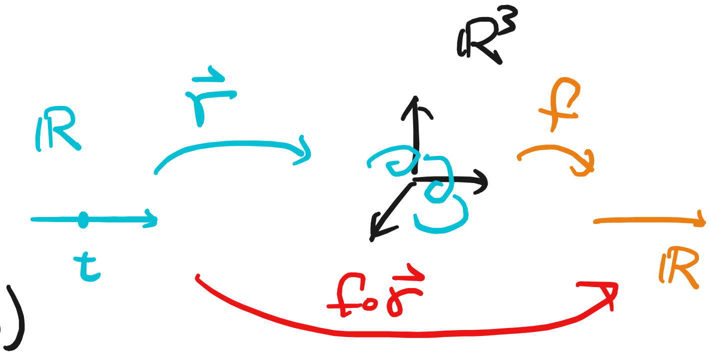
Calculus 5/14

Recall (Chain rule)

Let

$$f = f(x, y, z)$$

$$\vec{\gamma} = (\gamma_1(t), \gamma_2(t), \gamma_3(t))$$



be smooth functions. Then

$$\frac{d}{dt}\left(f(\vec{\gamma}(t))\right) = \nabla f(\vec{\gamma}(t)) \cdot \vec{\gamma}'(t)$$

$$= f_x(\vec{\gamma}(t))\,\gamma_1'(t) + f_y(\vec{\gamma}(t)) \cdot \gamma_2'(t) + f_z(\vec{\gamma}(t))\,\gamma_3'(t)$$

e.g. $f(x,y) = xy^2$, $\vec{\gamma}(t) = (\sin t, e^t)$

$\Rightarrow f_x = y^2$, $f_y = 2xy$, $\vec{\gamma}'(t) = (\cos t, e^t)$

$$\Rightarrow \frac{d}{dt}\left(f(\gamma_1(t), \gamma_2(t))\right) = (\underbrace{(e^t)^2}_{f_x(\vec{\gamma}(t))}, \underbrace{2\sin t \cdot e^t}_{f_y(\vec{\gamma}(t))}) \cdot (\cos t, e^t)$$

$$= e^{2t}\cos t + 2e^{2t}\sin t \quad \#$$

Apply Chain Rule to implicit functions

Example

Suppose y is a differentiable function of x that satisfies the equation

$$\underbrace{2x^2y - y^3 + 1 - x - 2y}_{= u(x,y)} = 0$$

Q: $\dfrac{dy}{dx} = ?$

$\vec{r}(x) = (x, y(x))$

<u>sol</u>:

$$0 = \frac{d}{dx}(0) = \frac{d}{dx}\left(u(\boxed{x,y})\right)$$

$$\overset{\text{chain rule}}{=} \left(u_x(x,y),\ u_y(x,y)\right) \bullet \left(1, \frac{dy}{dx}\right)$$

$u_x = 4xy - 1$,

$u_y = 2x^2 - 3y^2 - 2$

$$= (4xy-1)\cdot 1 + (2x^2-3y^2-2)\cdot\frac{dy}{dx}$$

$$= 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{4xy-1}{2x^2-3y^2-2} \qquad \#$$

<u>Example</u>

Assume $z = z(x,y)$ is a differentiable

function of (x,y). Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

with the condition:

$$\underbrace{z^4 + x^2 z^3 + y^2 + xy}_{u(x,y,z),\ z = z(x,y),\ \vec{\gamma}(x) \text{ or } \vec{\gamma}(y)} = 2$$

(i) $\Rightarrow 0 = \frac{\partial}{\partial x}(2) \overset{\text{chain}}{\underset{\text{rule}}{=}} u_x \cdot 1 + \cancel{u_y \cdot 0} + u_z \cdot \frac{\partial z}{\partial x}$

$$= (2xz^3 + y)\cdot 1 + (4z^3 + 3x^2z^2)\cdot \frac{\partial z}{\partial x}$$

$$\Rightarrow \frac{\partial z}{\partial x} = \frac{-(2xz^3+y)}{4z^3 + 3x^2z^2}$$

(ii) $0 = \frac{\partial}{\partial y}(2) \overset{\text{chain}}{\underset{\text{rule}}{=}} u_x \cdot 0 + u_y \cdot 1 + u_z \cdot \frac{\partial z}{\partial y}$

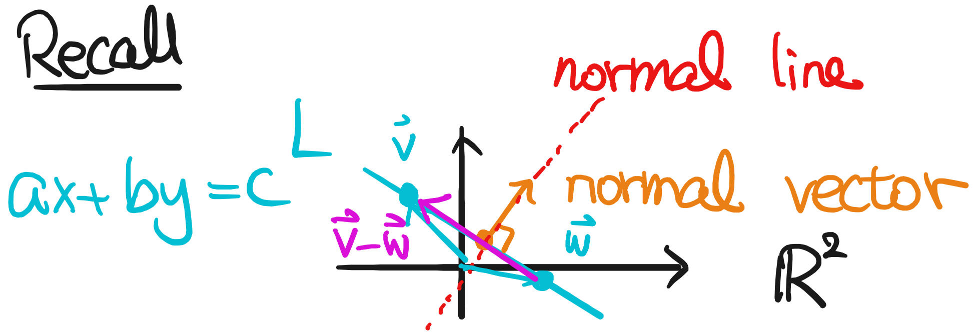
$$= (2y + x)\cdot 1 + (4z^3 + 3x^2z^2)\frac{\partial z}{\partial y}$$

$$\Rightarrow \frac{\partial z}{\partial y} = \frac{-(2y+x)}{4z^3 + 3x^2z^2}$$

#

Gradient and normal vector

Recall



A vector $\vec{n} \in \mathbb{R}^2$ is a normal vector of L iff

$$\vec{n} \bullet (\vec{v} - \vec{w}) = 0 \qquad \forall\, \overset{(v_1, v_2)}{\overset{\|}{\vec{v}}}, \underbrace{\vec{w}}_{(w_1, w_2)} \in L$$

Assume L is given by

$$ax + by = C$$

Then $(a, b) \bullet (x, y)$

$$(a, b) \bullet (\vec{v} - \vec{w})$$

$$= (a, b) \bullet (v_1 - w_1,\ v_2 - w_2)$$

$$= \underbrace{av_1 + bv_2}_{C} \; \underbrace{- aw_1 - bw_2}_{= -C}$$

$\vec{v} \in L \Leftrightarrow av_1 + bv_2 = C$

$$= C - C = 0$$

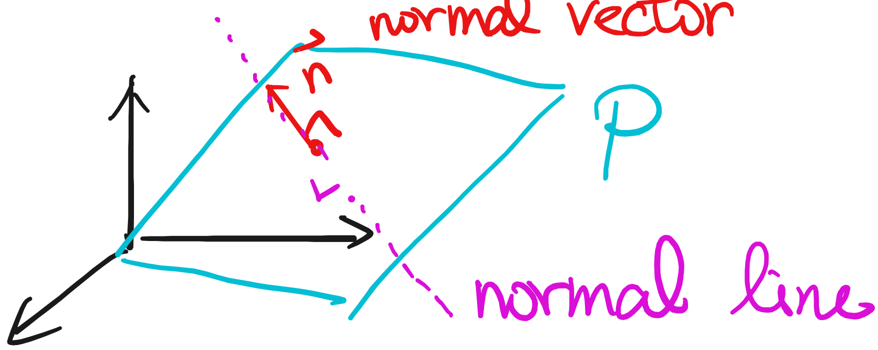
So $\vec{n} = (a, b)$ is a normal vector of L.

Similarly, a normal vector of a plane P in $\mathbb{R}^3$ is a vector

$\vec{n}$ that is perpendicular to P

i.e.

$$\vec{n} \bullet (\vec{v} - \vec{w}) = 0 \quad \forall\ \vec{v}, \vec{w} \in P$$



If P is given by

$$ax + by + cz = d,$$

then

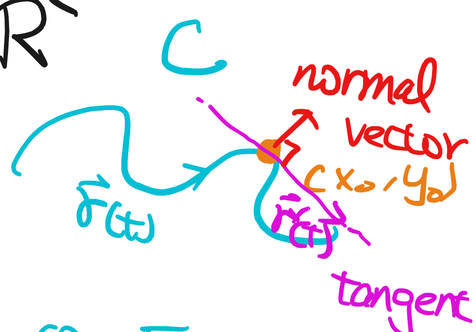
$$\vec{n} = (a, b, c)$$

is a normal vector of P.

- - - - -

Consider a curve in $\mathbb{R}^2$

$$C: \quad f(x, y) = c$$



Suppose $(x_0, y_0) \in C$, i.e.

$$f(x_0, y_0) = c$$

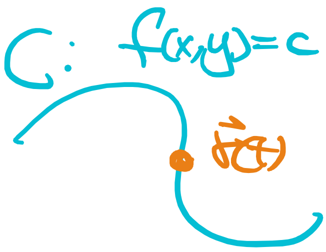
Q: Find a line normal vector of C at (x_0, y_0)

and $\nabla f(x_0, y_0) \neq 0 = (0,0)$.

Suppose, near (x_0, y_0), the curve C is paremetried by

$$\vec{\gamma}(t) = (\gamma_1(t), \gamma_2(t)),$$

$$\vec{\gamma}(0) = (x_0, y_0)$$



that is, $\vec{\gamma}(t) \in C$

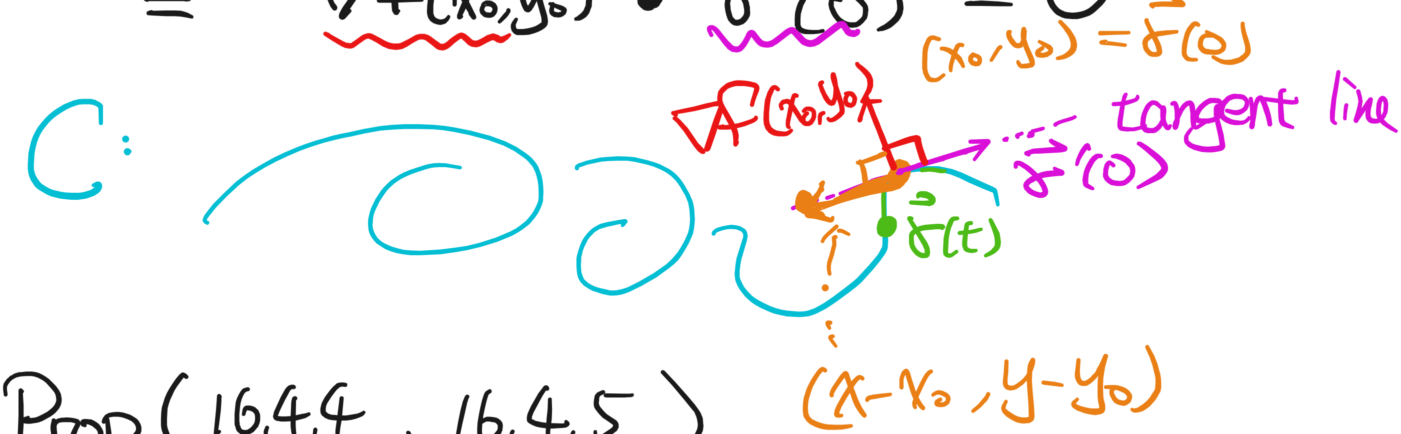
$$\underline{f(\gamma_1(t), \gamma_2(t)) = c}$$

$\Rightarrow$

$$0 = \frac{d}{dt}\Big|_{t=0}(c) = \frac{d}{dt}\Big|_{t=0}\Big(f(\gamma_1(t), \gamma_2(t))\Big)$$

$$\overset{\text{chain rule}}{=} \Big(f_x(\underbrace{\gamma_1(0), \gamma_2(0)}_{(x_0, y_0)}), f_y(\gamma_1(0), \gamma_2(0))\Big) \cdot \vec{\gamma}'(0)$$

$$= \nabla f(x_0, y_0) \cdot \vec{\gamma}'(0) = 0$$



Prop (16.4.4, 16.4.5)

Suppose $\nabla f(x_0, y_0) \neq \vec{0}$. The tangent line of the curve

$$C: f(x,y) = c$$

is determined by

$$\nabla f(x_0,y_0) \bullet (x - x_0,\ y - y_0) = 0$$

$$= f_x(x_0,y_0)\cdot(x-x_0) + f_y(x_0,y_0)\cdot(y-y_0) = 0$$

Furthermore, the normal line, i.e. the normal line of the tangent line, of C at (x_0,y_0) is determined by

$$\underset{\substack{\shortparallel \\ (f_x(x_0,y_0),\ f_y(x_0,y_0))}}{\nabla f(x_0,y_0)} \ /\!/ \ (x-x_0,\ y-y_0)$$

parallel

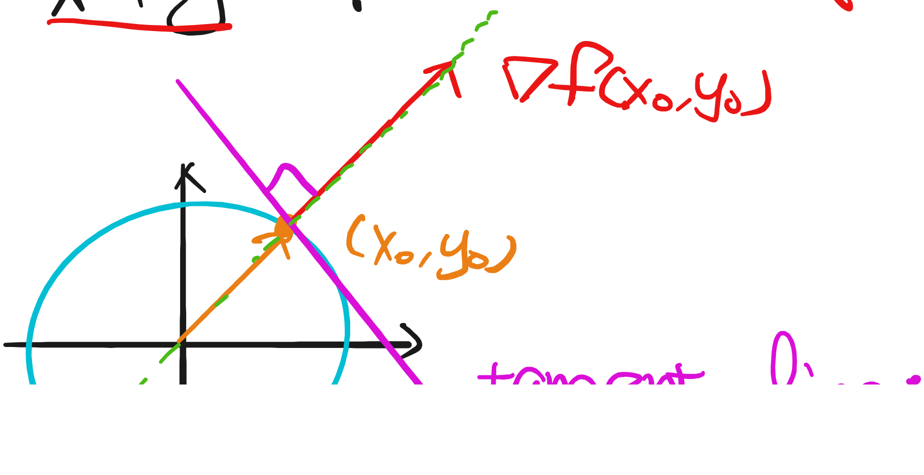
i.e.

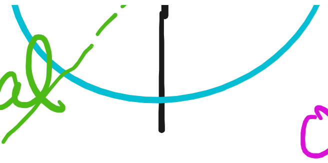
$$f_x(x_0,y_0)\cdot(y-y_0) = f_y(x_0,y_0)\cdot(x-x_0)$$

Example

Consider $\underbrace{x^2+y^2}_{f(x,y)} = 1$

$$\nabla f(x_0,y_0) = (2x_0,\ 2y_0)$$





tangent line:

$$0 = (2x_0, 2y_0) \bullet (x - x_0, y - y_0)$$

$$= 2x_0(x - x_0) + 2y_0(y - y_0)$$

normal line:

$$(2x_0, 2y_0) \,//\, (x - x_0, y - y_0)$$

$$x_0(y - y_0) = y_0(x - x_0)$$