Calculus 5/7

Recall

- Let $f = f(x,y,z)$ be smooth (i.e. all the partial derivatives of f of any order exist and are continuous)
- Under smoothness assumption,

$$f_{xy} = f_{yx}$$

- The gradient of f is

$$\nabla f = (f_x, f_y, f_z)$$
$$= f_x \vec{i} + f_y \vec{j} + f_z \vec{k}$$

Example

$(xy)z^2$

If $f(x,y,z) = xyz^2$, then

$$\nabla f(x,y,z) = (yz^2, xz^2, 2xyz)$$

$\overset{\text{(iii)}}{=} z^2 \nabla(xy) + (xy)\nabla(z^2) = z^2(y, x, 0) + xy(0, 0, 2z)$

$(f = xy,\ g = z^2)$

Thm (16.2.1)

Let f, g be (smooth) functions and $\alpha \in \mathbb{R}$.

Then

(i) $\nabla(f+g) = (\nabla f) + (\nabla g)$

(ii) $\nabla(\alpha f) = \alpha(\nabla f)$

→ (iii) $\nabla(fg) = g(\nabla f) + f(\nabla g)$

T

$$\nabla(fg) = \underbrace{(fg)_x}_{\text{product rule}} \vec{i} + (fg)_y \vec{j} + (fg)_z \vec{k}$$

$$= (f_x \cdot g + f \cdot g_x)\vec{i} + (f_y g + f g_y)\vec{j} + (f_z g + f \cdot g_z)\vec{k}$$

$$= g f_x \vec{i} + g f_y \vec{j} + g f_z \vec{k} + f g_x \vec{i} + f g_y \vec{j} + f g_z \vec{k}$$

$$= g \underbrace{(f_x \vec{i} + f_y \vec{j} + f_z \vec{k})}_{= \nabla f} + f \underbrace{(g_x \vec{i} + g_y \vec{j} + g_z \vec{k})}_{\nabla g}$$

$$= g \nabla f + f \nabla g \qquad \#$$

<u>Def</u> (Def 16.2.2)

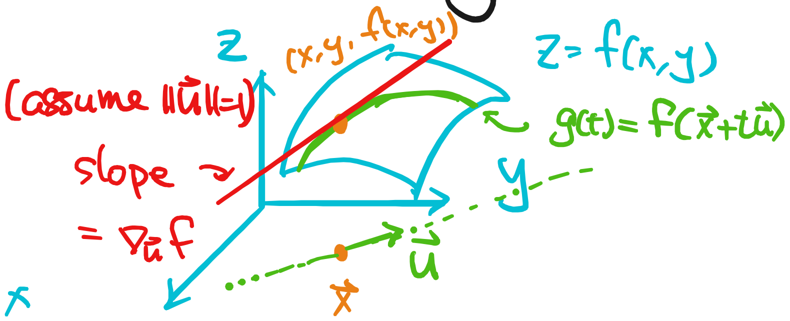
Let $\vec{u}$ be a vector.

The limit

$$\nabla_{\vec{u}} f(\vec{x}) = l \;\; \frac{f(\vec{x}+h\vec{u}) - f(\vec{x})}{\quad}$$

if it exists, is called the directional 方向導數 derivative of f at $\vec{x}$ along $\vec{u}$

$z = f(x,y)$



Remark

In the textbook, $\vec{u}$ is assumed to be a unit vector (i.e. $\|\vec{u}\|=1$)

單位向量 ⟷ "direction"

But this assumption is not very important.

Remark

$$\nabla_{\vec{i}} f(\vec{x}) = \lim_{h\to 0} \frac{f(\vec{x}+h\vec{i}) - f(\vec{x})}{h} = f_x(\vec{x})$$

$f(\vec{x}+h\vec{i}) = f(x+h, y, z)$

$$f_x(\vec{x}) = \lim_{h\to 0} \frac{f(x+h,y,z) - f(x,y,z)}{h}$$

$$\nabla_{\vec{j}} f = f_y$$

$$\nabla_{\vec{k}} f = f_z$$

a function of t

Remark

$$\nabla_{\vec{u}} f(\vec{x}) = \frac{d}{dt}\Big|_{t=0} f(\vec{x}+t\vec{u})$$

Thm (Thm 16.2.4)

$$\nabla_{\vec{u}} f(\vec{x}) = \nabla f(\vec{x}) \cdot \vec{u}$$

pf

We prove it for the case of two variables.

The case of n variables is similar.

Let $\vec{u} = u_1 \vec{i} + u_2 \vec{j}$,

$$\frac{f(\vec{x}+h\vec{u}) - f(\vec{x})}{h} = \frac{1}{h}\Big(f(x+hu_1, y+hu_2) - f(x,y)\Big)$$

$$= \frac{1}{h}\Big(f(\overset{b}{x+hu_1}, y+hu_2) - f(\overset{a}{x}, y+hu_2) + f(x, \overset{b}{y+hu_2}) - f(x, \overset{a}{y})\Big)$$

Recall (Mean Value Thm)

g : differentiable function of one variable on $h \in \mathbb{R}$

$\exists\, c$ between a and b s.t.

$$g'(c) = \frac{g(b) - g(a)}{b - a}$$

By Mean Value Thm,

$\exists\ c_h$ between $x + hu_1$ and x, and

c_h' between $y + hu_2$ and y

s.t.

$g(t) = f(t, y+hu_2)$

$$\frac{f(x+hu_1, y+hu_2) - f(x, y+hu_2)}{\cancel{x}+hu_1 - \cancel{x}} = f_x(c_h, y+hu_2)$$

and

$g(t) = f(x, t)$

$$\frac{f(x, y+hu_2) - f(x,y)}{\cancel{y}+hu_2 - \cancel{y}} = f_y(x, c_h')$$

So

$$\nabla_{\vec{u}} f(x,y) = \lim_{h\to 0} \frac{f(\vec{x}+h\vec{u}) - f(\vec{x})}{h}$$

$$= \lim_{h\to 0} \frac{1}{\cancel{h}}\Big(\underbrace{(f(x+hu_1, y+hu_2) - f(x, y+hu_2))}_{= \cancel{h}u_1 \cdot f_x(c_h,\, y+hu_2)} + \underbrace{(f(x, y+hu_2) - f(x,y))}_{\cancel{h}u_2\, f_y(x,\, c_h')} \Big)$$

$$= \lim_{h\to 0} \Big(u_1 \cdot f_x(c_h, y+hu_2) + u_2 \cdot f_y(x, c_h') \Big)$$

$x \quad y \quad y$

$h \to 0$

$\because f_x$ and f_y are continuous

$$= u_1 \cdot f_x(x,y) + u_2 \cdot f_y(x,y)$$

$$= \nabla f(x,y) \bullet \vec{u} \qquad \#$$

<u>Example</u>

Let

$$f(x,y) = x^2 + y^2$$

$$\vec{u} = \frac{3}{5}\vec{i} + \frac{4}{5}\vec{j}$$

Q: $\quad \nabla_{\vec{u}} f(1,2) = ?$

<u>sol</u>

$$\nabla f = 2x\vec{i} + 2y\vec{j}$$

$$\nabla f(1,2) = 2\vec{i} + 4\vec{j}$$

$$\Rightarrow \quad \nabla_{\vec{u}} f(1,2) = (2\vec{i} + 4\vec{j}) \bullet \left(\frac{3}{5}\vec{i} + \frac{4}{5}\vec{j}\right)$$

$$= 2 \cdot \frac{3}{5} + 4 \cdot \frac{4}{5} = \frac{6}{5} + \frac{16}{5} = \frac{22}{5} \quad \#$$

<u>Example</u>

Consider

$g(t) = f(x + tu)$

Q: $g'(t) = ?$ (We know $g'(0) = \nabla_{\vec{u}} f(\vec{x})$)

<u>Sol</u>

$$g'(t) = \lim_{h \to 0} \frac{g(t+h) - g(t)}{h}$$

$$= \frac{d}{ds}\Big|_{s=0} g(t+s) = \frac{d}{ds}\Big|_{s=0} f(\vec{x} + (t+s)\vec{u})$$

$$= \frac{d}{ds}\Big|_{s=0} f(\underbrace{(\vec{x} + t\vec{u})}_{\vec{y}} + s\vec{u})$$

$$= \nabla_{\vec{u}} f(\vec{y})$$

$$= \nabla f(\vec{y}) \cdot \vec{u}$$

$$= \nabla f(\vec{x} + t\vec{u}) \cdot \vec{u} \qquad \#$$

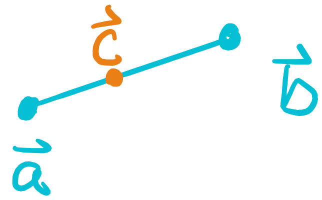
<u>Notation</u>:

In textbook,

$$\underbrace{f'_{\vec{u}}}_{} = \nabla_{\vec{u}} f$$

<u>Thm</u> (Thm 16.3.1, Mean Value Thm for Functions of several variables)

Let f be a smooth function.



There exists

$\vec{c} \in$ the line segment connecting $\vec{a}$ and $\vec{b}$

s.t.

$$f(\vec{b}) - f(\vec{a}) = \nabla f(\vec{c}) \bullet (\vec{b} - \vec{a})$$