Calculus 5/2

Recall (Partial derivative)

$$f(x,y) = xy^2$$

$$\Rightarrow \quad f_x(x,y) = \frac{\partial f}{\partial x} = y^2$$

$$f_y(x,y) = \frac{\partial f}{\partial y} = 2xy$$

Def (can be more than 3)

Let f be a function of three variables.

$$f = f(x,y,z)$$

Then

$$f_x = \frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h,y,z) - f(x,y,z)}{h}$$

$$f_y = \frac{\partial f}{\partial y} = \lim_{h\to 0} \frac{f(x,y+h,z) - f(x,y,z)}{h}$$

$$f_z = \frac{\partial f}{\partial z} = \lim_{h\to 0} \frac{f(x,y,z+h) - f(x,y,z)}{h}$$

(Assume the limits exist)

If

$$f = f(x_1, \dots, x_n)$$

then

$$f_{x_k} = \frac{\partial f}{\partial x_k} = \lim_{h\to 0} \frac{f(x_1, \cdots x_{k-1}, x_k + h, x_{k+1}, \cdots, x_n) - f(x_1, \cdots, x_n)}{h}$$

Example

① $f(x,y,z) = xy^2z^3$

$\Rightarrow f_x = y^2 z^3, \quad \Rightarrow f_x(0,1,2) = 1^2 \cdot 2^3 = 8$

$f_y = 2xyz^3$

$f_z = 3xy^2z^2$

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② $f(x,y,z) = x^2 e^{\frac{y}{z}}, \quad z \neq 0$

$\Rightarrow f_x = 2x e^{\frac{y}{z}}$

$f_y = x^2 e^{\frac{y}{z}} \cdot \frac{\partial (\frac{y}{z})}{\partial y} = \frac{x^2}{z} \cdot e^{\frac{y}{z}}$

$f_z = x^2 e^{\frac{y}{z}} \cdot \frac{\partial}{\partial z}\left(\frac{y}{z}\right) = -\frac{yx^2}{z^2} e^{\frac{y}{z}}$

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Higher order partial derivative and Continuity

$f = f(x,y,z)$

1st order partial derivatives:

f_x, f_y, f_z

2nd order partial derivatives:

$$\frac{\partial(f_x)}{\partial x} = f_{xx} = \frac{\partial^2 f}{\partial x^2}, \quad f_{yx}, \quad f_{zx}$$

$$\frac{\partial(f_x)}{\partial y} = f_{xy} = \frac{\partial^2 f}{\partial y \partial x}, \quad f_{yy}, \quad f_{zy}$$

$$\frac{\partial(f_x)}{\partial z} = f_{xz} = \frac{\partial^2 f}{\partial z \partial x}, \quad f_{yz}, \quad f_{zz}$$

3rd order:

$$f_{xxx} = \frac{\partial^3 f}{\partial x^3}, \quad f_{xxy} = \frac{\partial^3 f}{\partial y \partial x^2}, \quad \cdots$$

In general,

k-th order partial derivative

= a partial derivative of a (k-1)-th order partial derivative

<u>Example</u>

$$f(x,y,z) = x y^2 z^3$$

1st order:

$$f_x = y^2z^3, \quad f_y = 2xyz^3, \quad f_z = 3xy^2z^2$$

2nd order:

$$f_{xx} = (f_x)_x = 0, \quad f_{xy} = 2yz^3, \quad f_{xz} = 3y^2z^2$$

$$f_{yx} = 2yz^3, \quad f_{yy} = 2xz^3, \quad \boxed{f_{yz} = 6xyz^2},$$

$$f_{zx} = 3y^2z^2, \quad \boxed{f_{zy} = 6xyz^2}, \quad f_{zz} = 6xy^2z$$

Observation

$$f_{xy} = f_{yx} \ ??$$

In fact, this observation is true under some assumptions

To describe the assumptions, we need "continuity"

For simplicity, we consider

$$f = f(x,y).$$

Def (§15.6)

Let $f = f(x,y)$.

$\vec{x} = (x,y)$, $\vec{a} = (a_1, a_2) \in \mathbb{R}^2$.

We say the limit

$$\lim_{\vec{x} \to \vec{a}} f(\vec{x}) \quad \text{exists}$$

if there is a number $L \in \mathbb{R}$ s.t.

$\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.

if $0 < \|\vec{x} - \vec{a}\| < \delta$, then

$$|f(\vec{x}) - L| < \varepsilon$$

In this case, we write

$$\lim_{\vec{x} \to \vec{a}} f(\vec{x}) = L$$

A function $f(\vec{x})$ is continuous at $\vec{a} = (a_1, a_2)$ if

$$\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = f(\vec{a}) = f(a_1, a_2)$$

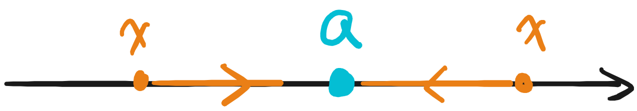
We say that $f(\vec{x})$ is continuous if $f(\vec{x})$ is continuous at each point $\vec{a}$ in $\mathbb{R}^2$.

Remark

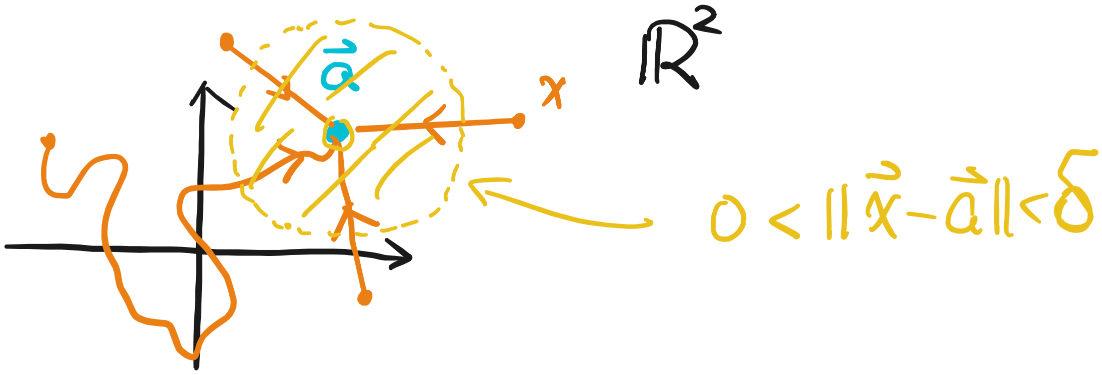
Consider

$$\vec{x} \longrightarrow \vec{a}$$

① one variable :



② more than one variables:



Facts:

The following functions are continuous on $\mathbb{R}^2$

① Constant functions.

② polynomial of two (or many) variables:

x, y, $x+y$, xy, x^4y+y^3-x+1, etc.

③ $e^{P(x,y)}$, $\cos P(x,y)$, $\sin P(x,y)$ and

(algebraic)
"linear combinations" of them

e.g. $1\cdot e^{xy} + 2\cdot\cos x + (-3)\sin(y^2)$

e.g.

e^{xy}, $\cos(x^2+y)$, $\sin(e^x + (\cos(xy))\cdot e^{y^2})$

etc.

Thm (p. 783)

Let $f = f(x,y)$. Suppose that

f, f_x, f_y, f_{xy}, f_{yx}

are continuous on $\mathbb{R}^2$. Then

$$f_{xy} = f_{yx} \quad \text{on } \mathbb{R}^2$$

For convenience, we will assume that functions are smooth unless otherwise stated.

Here,

smooth = all partial derivatives of any order exist and continuous

Note: the functions mentioned above are smooth:

① Constant functions.

② polynomial of two (or many) variables:

$x,\ y,\ x+y,\ xy,\ x^4y+y^3-x+1$, etc.

③ $e^{P(x,y)}$, $\cos P(x,y)$, $\sin P(x,y)$ and

"(algebraic) linear combinations" of them

e.g. $1\cdot e^{xy} + 2\cdot\cos x + (-3)\sin(y^2)$

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etc.

Gradient

<u>Def</u> (Def 16.1.2 and Thm 16.1.3)

Let $f = f(x,y,z)$.

Then gradient of f at $\vec{x} \in \mathbb{R}^3$ is
$\vec{x} = (x,y,z)$

the vector

$$\nabla f(\vec{x}) = \frac{\partial f}{\partial x}(\vec{x})\cdot\vec{i} + \frac{\partial f}{\partial y}(\vec{x})\vec{j} + \frac{\partial f}{\partial z}(\vec{x})\vec{k}$$

$$= \left(\frac{\partial f}{\partial x}(\vec{x}),\ \frac{\partial f}{\partial y}(\vec{x}),\ \frac{\partial f}{\partial z}(\vec{x})\right)$$

If $f = f(x,y)$, then

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$$

<u>Example</u>

① $f(x,y) = xe^y - ye^x$.

$\Rightarrow$ $f_x = e^y - ye^x$, $\quad f_y = xe^y - e^x$

$\Rightarrow$

$$\nabla f(x,y) = (e^{y} - ye^{x})\vec{i} + (xe^{y} - e^{x})\vec{j} \quad \#$$

② $f(x,y,z) = x\sin(\pi y) + y\cos(\pi z)$

$\nabla f(0,1,2) = ?$

Sol

$f_x = \sin(\pi y), \quad f_y = \pi x\cos(\pi y) + \cos(\pi z)$

$f_z = -\pi y\sin(\pi z)$

$$\Rightarrow f_x(0,1,2) = \sin(\pi\cdot 1) = 0$$

$$f_y(0,1,2) = \overbrace{\cancel{\pi\cdot 0\cos(\pi\cdot 1)}}^{=0} + \cos(\pi\cdot 2) = 1$$

$$f_z(0,1,z) = -\pi\cdot 1\sin(\pi\cdot 2) = 0$$

So

$$\nabla f(0,1,2) = 0\cdot\vec{i} + 1\vec{j} + 0\cdot\vec{k} = \vec{j}$$

$$= (0,1,0) \quad \#$$

Remark

Our definition is different from the definition (Def 16.1.2) in the textbook.

The two definitions are actually equivalnt for smooth functions.

(Thm 16.1.3)