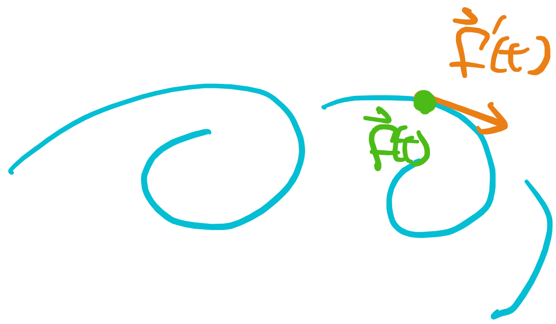
# Calculus 4/30

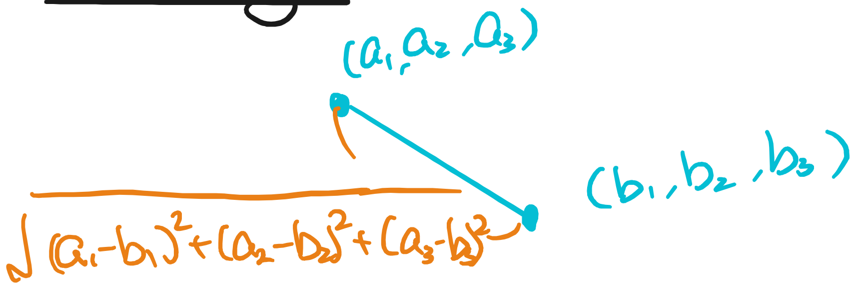
Recall

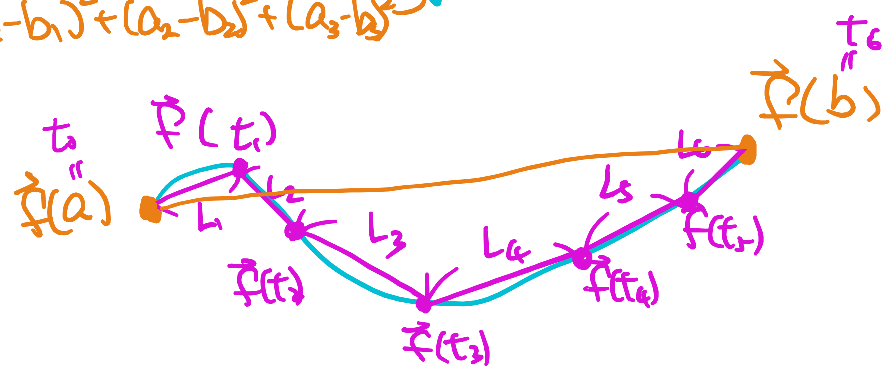
If $\vec{f}(t) = (f_1(t), f_2(t), f_3(t))$,

then $\vec{f}'(t) = (f_1'(t), f_2'(t), f_3'(t))$




Arc length




A partition of $[a, b]$:

$$P = \{a = t_0 < t_1 < \cdots < t_6 = b\}$$

$$\text{length}(\sim) \approx L_1 + L_2 + \cdots + L_6$$

$$= \lim_{\|P\| \to 0} \sum_{i=1}^{n} \underline{\|\vec{f}(t_{i+1}) - \vec{f}(t_i)\|}$$

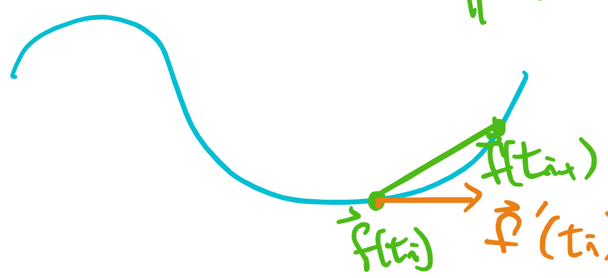
c.f.:

Riemann sum

" $P=\{a=t_0<t_1<\cdots<t_n=b\}$ "  (integration)

$\| (t_{i+1}-t_i)\cdot \vec{f}'(t_i)\|$ "$\sum f(c_i)(x_{i+1}-x_i)$"

$= \|\vec{f}'(t_i)\|\cdot(t_{i+1}-t_i)$




$$\vec{f}'(t_i)=\lim_{h\to 0}\frac{\vec{f}(t_i+h)-\vec{f}(t_i)}{h}$$

$h=t_{i+1}-t_i$

$$\approx \frac{\vec{f}(t_{i+1})-\vec{f}(t_i)}{t_{i+1}-t_i}$$

**Thm** (Thm 14.4.2)

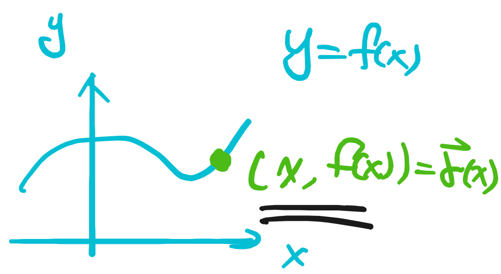
Let $\vec{f}(t)$, $t\in[a,b]$, be a continuously differentiable parametrized curve in $\mathbb{R}^3$. Then the arc length $L$ of $\vec{f}(t)$, $t\in[a,b]$, is given by

$$L=\int_a^b \|\vec{f}'(t)\|\,dt$$

**Remark**

Consider

$$y=f(x)$$




$\Rightarrow$

$$L=\int_a^b \|\vec{\gamma}'(x)\|\,dx$$

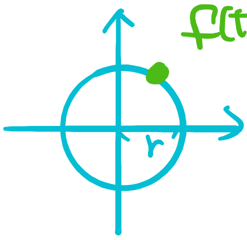
$$=\int^b \|(1, f'(x))\|\,dx$$

cf. the arc length

formula in fall

$$= \int_a^b \sqrt{1 + (f'(x))^2}\, dx \qquad \#$$

**Example**




$\vec{f}(t) = (r\cos t,\ r\sin t),\quad 0 \le t \le 2\pi$

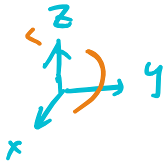
$$L = \int_0^{2\pi} \| \vec{f}'(t) \|\, dt$$

$$= \int_0^{2\pi} \| (r(-\sin t),\ r\cos t) \|\, dt$$

$$= \int_0^{2\pi} \sqrt{r^2 \underbrace{\sin^2 t + r^2 \cos^2 t}_{1}}\, dt$$

$$= \int_0^{2\pi} r\, dt = 2\pi r \qquad \#$$

**Example**




$$\vec{f}(t) = (\cos t,\ \sin t,\ t)$$

$0 \le t \le \frac{\pi}{2}$.

$$\text{Arc length} = \int_0^{\frac{\pi}{2}} \| \vec{f}'(t) \|\, dt$$

$$= \int_0^{\frac{\pi}{2}} \| (-\sin t,\ \cos t,\ 1) \|\, dt$$

$$= \int_0^{\frac{\pi}{2}} \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2}\, dt$$

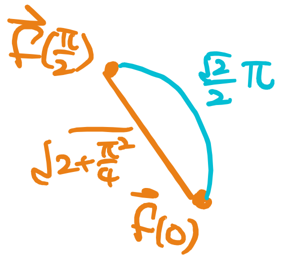
$$= \int_0^{\frac{\pi}{2}} \sqrt{2}\, dt = \frac{\sqrt{2}}{2}\pi \qquad \#$$

Note:

$\vec{f}(0) = (1,0,0)$

$\vec{f}(\frac{\pi}{2}) = (0,1,\frac{\pi}{2})$

$\|\vec{f}(0)-\vec{f}(\frac{\pi}{2})\| = \sqrt{1^2+1^2+(\frac{\pi}{2})^2} = \sqrt{2+\frac{\pi^2}{4}} < \frac{\sqrt{2}}{2}\pi$




$\sqrt{\frac{\pi^2}{2}}$ $\|$

## § Functions of several variables

Consider real-valued functions of several variables, e.g.

$$f(x,y) = x^2+y^2$$

$$g(x,y,z) = x^2+zy^2+xz^3y$$

### Partial derivative 偏微分

Def (Def 15.4.1)

Let

$$f = f(x,y).$$

The partial derivative of $f$ with repect to $x$

" " " $y$

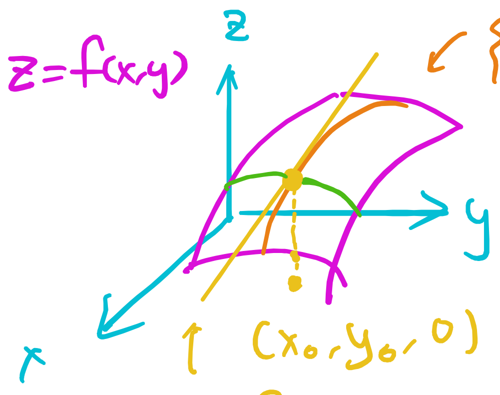
are

$$f_x(x,y) = \frac{\partial f}{\partial x} = \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}$$

$$f_y(x,y) = \frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

if the limits exist.

$\{z=f(x,y)\} \cap \{y=y_0\}$




slope of tangent line of ⌒ at $(x_0, y_0, f(x_0,y_0))$

$$= \frac{\partial f}{\partial x}(x_0, y_0)$$

## Example

Calculate $f_x$ and $f_y$:

① $f(x,y) = x^2 + y^2$:

$$\frac{\partial f}{\partial x} = \frac{\partial x^2}{\partial x} + \frac{\partial y^2}{\partial x} = 2x + 0 = 2x$$

$$\frac{\partial f}{\partial y} = \frac{\partial x^2}{\partial y} + \frac{\partial y^2}{\partial y} = 0 + 2y = 2y \quad \#$$

② $f(x,y) = x \cdot \tan^{-1}(xy)$ product rule

$$\frac{\partial f}{\partial x} = \frac{\partial (x \cdot \tan^{-1}(xy))}{\partial x} = \frac{\partial x}{\partial x} \cdot \tan^{-1}(xy)$$

Recall

$\checkmark \quad \frac{1}{\quad} \cdot \frac{\partial(xy)}{}$

$$+ x \cdot \left( \partial \tan^{-1}(xy) \right.$$

$(\tan^{-1}(x))' = \frac{1}{1+x^2}$ $\quad 1+(xy)^2 \quad \partial x$ $\quad \xrightarrow{\text{chain rule}} \quad \frac{\quad}{\partial x}$

$$= \tan^{-1}(xy) + x \cdot \frac{y}{1+(xy)^2}$$

$$\frac{\partial f}{\partial y} = \frac{\partial\,(x \cdot \tan^{-1}(xy))}{\partial y} = x \cdot \frac{\partial(\tan^{-1}(xy))}{\partial y}$$

$$= x \cdot \frac{1}{1+(xy)^2} \cdot \underbrace{\frac{\partial(xy)}{\partial y}}_{= x} = \frac{x^2}{1+(xy)^2} \quad \#$$

③ $f(x,y) = e^{xy} + \ln(x^2+y)$

$$\frac{\partial f}{\partial x} = \frac{\partial e^{xy}}{\partial x} + \frac{\partial}{\partial x}(\ln(x^2+y))$$

$$= e^{xy} \cdot \underbrace{\frac{\partial(xy)}{\partial x}}_{= y} + \frac{1}{x^2+y} \cdot \underbrace{\frac{\partial(x^2+y)}{\partial x}}_{2x}$$

$$= ye^{xy} + \frac{2x}{x^2+y}$$

$$\frac{\partial f}{\partial y} = \frac{\partial e^{xy}}{\partial y} + \frac{\partial}{\partial y}(\ln(x^2+y))$$

$$= e^{xy} \cdot \underbrace{\frac{\partial(xy)}{\partial y}}_{= x} + \frac{1}{x^2+y} \underbrace{\frac{\partial(x^2+y)}{\partial y}}_{= 1}$$

$$= xe^{xy} + \frac{1}{x^2+y} \quad \#$$

# National Tsing Hua University

# Calculus II – Exam 2

Instructor: Hsuan-Yi Liao

Spring, 2024

**Name:** Answer

**Student ID:** ____________________

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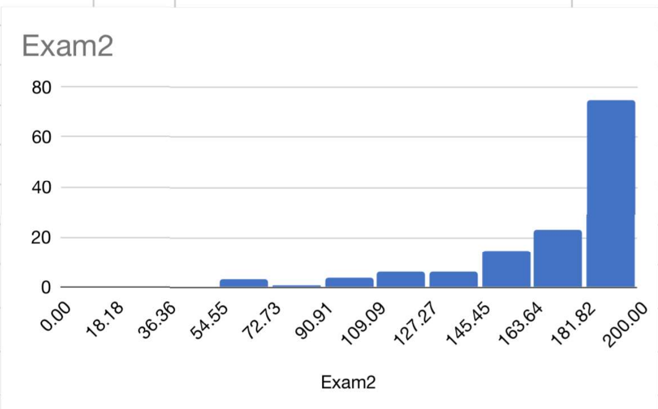
- This exam contains 8 pages (including this cover page) and 9 questions.
- Total of points is 200.
- Time limit: **100 minutes**.
- Write down your computation or arguments in details unless otherwise stated.
- The use of a calculator, cell phone, or any other electronic device is **NOT** permitted.
- The use of books or notes of any kind is **NOT** permitted.

**Distribution of Marks**

| Question | Points | Score |
|---|---|---|
| 1 | 8 | |

| | | | | | |
|---|---|---|---|---|---|
| 162 | Average | 299.48 | ~~138.414~~ | 174.290 | #[ |
| 163 | Average (except ze | 305.591 | ~~138.414~~ | 174.290 | #[ |
| 164 | Quartile 0 | 0 | ~~36~~ | 59 | #I |
| 165 | Quartile 1 | 273.25 | ~~117~~ | 164 | #I |
| 166 | Quartile 2 | 325 | ~~142~~ | 186 | #I |
| 167 | Quartile 3 | 356 | ~~162.5~~ | 195 | #I |
| 168 | Quartile 4 | 396 | ~~196~~ | 200 | #I |
| 169 | 標準差 | 81.5494 | ~~31.2913~~ | 30.3226 | #[ |
| 170 | 非零數 | 147 | ~~147~~ | 141 | |
| 171 | 不到60%人數 | 21 | ~~39~~ | 10 | |
| 172 | 滿分人數 | 0 | ~~0~~ | 8 | |




| Total: | 200 | |
|---|---|---|

1. Determine whether the series converges or diverges. No need to explain.

C (a) (2 points) $\sum_{k=2}^{\infty} \frac{k}{k^3 - k}$.

C (b) (2 points) $\sum_{k=1}^{\infty} \frac{k^k}{3^{k^2}}$.

C (c) (2 points) $\sum_{k=1}^{\infty} \frac{2 \cdot 4 \cdots 2k}{(2k)!}$.

D (d) (2 points) $\sum_{k=1}^{\infty} \frac{k!}{k^{k/2}}$.

2. Let $\vec{a} = (1, 2, 3)$, $\vec{b} = (-1, 0, 1)$ and $\vec{c} = (0, 0, 1)$. Compute the following.

(a) (4 points) $3(\vec{a} + \vec{b} - 2\vec{c})$. $= 3\,(1-1+0,\ 2+0-0,\ 3+1-2) = (0,6,6)$

(b) (6 points) $\|\vec{a} - \vec{c}\| = \|(1,2,2)\| = \sqrt{1+4+4} = 3$

(c) (6 points) $(\vec{a} + \vec{c}) \cdot (\vec{b} - \vec{c})$.

$= (1,2,4) \cdot (-1,0,0) = -1$

3. Is the series absolutely convergent, conditionally convergent or divergent? Explain your answer.

(a) (8 points) $\sum_{k=1}^{\infty}\left(\frac{(-1)^k}{k}-\frac{2}{k!}\right)$.

(b) (8 points) $\sum_{k=1}^{\infty}(-1)^k k\sin(1/k)$.

(c) (8 points) $\sum_{k=1}^{\infty}\frac{\sin(\pi k/2)}{k\sqrt{k}}$.

(a) Conditionally convergent, because $\sum \frac{(-1)^k}{k}$ and $\sum \frac{2}{k!}$ converge but $\left|\frac{(-1)^k}{k}-\frac{2}{k!}\right| \geq \frac{1}{k}-\frac{2}{k!}$ and $\sum \frac{1}{k}-\frac{2}{k!}$ diverges #

$\boxed{\sum (-1)^k a_k \text{ converges if } a_k \geq 0,\ a_k \searrow 0 \quad \lim_{k\to\infty} a_k = 0 \quad a_k \text{ is decreasing}}$

(b) $\lim_{k\to\infty} k\sin(\frac{1}{k}) = \lim_{k\to\infty}\frac{\sin(\frac{1}{k})}{\frac{1}{k}} = 1 \neq 0$

$\frac{\sin x}{x} \to 1$ as $x \to 0$

$\Rightarrow \lim_{k\to\infty}(-1)^k k\sin(\frac{1}{k}) \neq 0 \Rightarrow$ diverges #

$\boxed{\sum a_k \text{ converges} \Rightarrow \lim_{k\to\infty} a_k = 0}$

(c) $\left|\frac{\sin(\frac{\pi}{2}k)}{k\sqrt{k}}\right| \leq \frac{1}{k^{\frac{3}{2}}}$ and $\sum \frac{1}{k^{\frac{3}{2}}}$ converges

$\Rightarrow \sum \frac{\sin(\frac{\pi}{2}k)}{k\sqrt{k}}$ absolutely converges. #

4. Let $f$ be a function which can be differentiated infinitely many times on $(-1,1)$. The $n$**-th Taylor polynomial** of $f(x)$ is

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n.$$

The $n$**-th remainder** of $f(x)$ is

$$R_n(x) = f(x) - P_n(x).$$

(a) (16 points) Prove that

$$R_2(x) = \frac{1}{2}\int_0^x f^{(3)}(t)\cdot(x-t)^2 dt$$

for each $x \in (-1,1)$.

(b) (8 points) Find ~~$R_2$~~ $P_2$ for $f(x) = \sqrt{1+x}$.

(c) (8 points) Show that if $f(x) = \sqrt{1+x}$, then

$$|R_2(x)| < \frac{\sqrt{2}}{32}, \qquad \forall\, x \in (-1/2, 1/2).$$

(a) $$f(x) = f(0) + \int_0^x f'(t)\,dt = f(0) + \Big( \underbrace{f'(t)\cdot(t-x)\Big|_0^x}_{=f'(0)\cdot x} + \int_0^x f''(t)\,(-t+x)\,dt \Big)$$

$$= f(0) + f'(0)x + \Big( \underbrace{f''(t)\,\frac{-(x-t)^2}{2}\Big|_0^x}_{=\frac{f''(0)}{2}x^2} + \int_0^x f'''(t)\,\frac{(x-t)^2}{2}\,dt \Big)$$

$$= \underbrace{f(0) + f'(0)x + \frac{f''(0)}{2}x^2}_{P_2(x)} + \frac{1}{2}\int_0^x f'''(t)\cdot(x-t)^2\,dt$$

$$\Rightarrow R_2(x) = \frac{1}{2}\int_0^x f'''(t)\cdot(x-t)^2\,dt$$

(b) $f(0)=1$, $f'(0) = \frac{1}{2}(1+x)^{-\frac{1}{2}}\Big|_{x=0} = \frac{1}{2}$, $f''(0) = -\frac{1}{4}(1+x)^{-\frac{3}{2}}\Big|_{x=0} = -\frac{1}{4}$

$P_2(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2$ #

(c) $R_2(x) = \frac{1}{3!}f^{(3)}(c)\cdot x^3$ for some $c \in (-\frac{1}{2}, \frac{1}{2})$

$$= \frac{1}{3!}\cdot\frac{3}{8}(1+c)^{-\frac{5}{2}}\cdot x^3$$

$$\Rightarrow |R_2(x)| \leq \frac{1}{16}\cdot\underbrace{\left(\frac{1}{2}\right)^{-\frac{5}{2}}}_{2^{\frac{5}{2}}=4\sqrt{2}}\cdot\left|\frac{1}{2}\right|^3 = \frac{\sqrt{2}}{32} \quad \#$$

$\forall\, x \in (-\frac{1}{2}, \frac{1}{2})$

5. Find the interval of convergence.

(a) (12 points) $\sum_{k=1}^{\infty} kx^k$.

(b) (12 points) $\sum_{k=1}^{\infty} \frac{1}{(2k)!} x^{4k}$.

(c) (12 points) $\sum_{k=1}^{\infty} \left(1+\frac{1}{k}\right)^k (x+2)^k$.

$\sum a_k x^k$

(a) $\lim_{k\to\infty} \sqrt[k]{k} = 1 \Rightarrow r = $ radius of convergence $= 1$

At $x=1$, $\sum_{k=1}^{\infty} k$ diverges

At $x=-1$, $\sum_{k=1}^{\infty} (-1)^k k$ also diverges

So ans $= (-1, 1)$ #

(b) Since $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ Converges $\forall x$

$\cosh x = \frac{e^x + e^{-x}}{2} = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}$ also converges $\forall x$

$\Rightarrow \cosh(x^2) - 1 = \sum_{k=1}^{\infty} \frac{x^{4k}}{(2k)!}$ also Converges $\forall x$

$\Rightarrow$ interval of convergence $= (-\infty, \infty)$ #

(c) $\lim_{k\to\infty} \sqrt[k]{(1+\frac{1}{k})^k} = 1 \Rightarrow r = 1$

$|x+2| < 1$

$\Leftrightarrow -1 < x+2 < 1$

$\Leftrightarrow -3 < x < -1$

At $x = -1$, $\sum_{k=1}^{\infty} (1+\frac{1}{k})^k$ diverges because $\lim_{k\to\infty} (1+\frac{1}{k})^k = e \neq 0$

$e \neq 0$

At $x = -3$, $\sum_{k=1}^{\infty} \boxed{(1+\frac{1}{k})^k} \cdot (-1)^k$ diverges because $\boxed{\lim_{k\to\infty} (-1)^k (1+\frac{1}{k})^k \text{ diverges}}$ $\neq 0$

So ans $= (-3, -1)$ #

6. Expand $f(x)$ in powers of $x$.

   (a) (12 points) $f(x) = \dfrac{1}{(1-x)^2}$. $= \left(\frac{1}{1-x}\right)' = \left(\sum_{k=0}^{\infty} x^k\right)' = \sum_{k=1}^{\infty} k x^{k-1} = \sum_{k=0}^{\infty} (k+1)\, x^k$ #

   (b) (12 points) $f(x) = x^2 \arctan x$.

   (c) (12 points) $f(x) = \dfrac{1}{1-x} + e^{2x^3}$.

(b) $(\tan^{-1} x)' = \dfrac{1}{1+x^2} = \sum_{k=0}^{\infty} (-1)^k x^{2k}$

$\Rightarrow \tan^{-1} x = \sum_{k=0}^{\infty} (-1)^k \dfrac{x^{2k+1}}{2k+1} + C$ $\quad \tan^{-1} 0 = 0 \Rightarrow C = 0$

$\Rightarrow x^2 \tan^{-1} x = \sum_{k=0}^{\infty} (-1)^k \dfrac{x^{2k+3}}{2k+1}$ #

(c) $f(x) = \sum_{k=0}^{\infty} x^k + \sum_{k=0}^{\infty} \dfrac{1}{k!} (2x^3)^k$

$= \sum_{k=0}^{\infty} x^k + \sum_{k=0}^{\infty} \dfrac{2^k}{k!} x^{3k}$ #

7. Set $f(x) = \dfrac{e^x - 1}{x}$.

   (a) (8 points) Expand $f(x)$ in a power series.

   (b) (16 points) Differentiate the series and show that

$$\sum_{n=1}^{\infty} \frac{n}{(n+1)!} = 1.$$

$$f(x) = \frac{\sum_{k=1}^{\infty} \frac{x^k}{k!}}{x} = \sum_{k=1}^{\infty} \frac{x^{k-1}}{k!} = \sum_{k=0}^{\infty} \frac{x^k}{(k+1)!} \quad \forall x \in (-\infty, \infty)$$

(b)
$$f'(x) = \sum_{k=1}^{\infty} \frac{k}{(k+1)!} x^{k-1} = \left(\frac{e^x - 1}{x}\right)' = \frac{e^x \cdot x - e^x + 1}{x^2} \quad \forall x \in (-\infty, \infty)$$

$$\Rightarrow f'(1) = e - e + 1 = 1 = \sum_{k=1}^{\infty} \frac{k}{(k+1)!}$$

#

8. Given that

$$\|\vec{a}\| = 1, \quad \|\vec{b}\| = 3, \quad \|\vec{c}\| = 4, \quad \vec{a}\cdot\vec{b} = 0, \quad \vec{a}\cdot\vec{c} = 1, \quad \vec{b}\cdot\vec{c} = -2,$$

find

(a) (6 points) $3\vec{a}\cdot(\vec{b}+4\vec{c})$. $= 3\vec{a}\cdot\vec{b} + 12\,\vec{a}\cdot\vec{c} = 12$ #

(b) (6 points) $(\vec{a}-\vec{b})\cdot(2\vec{a}+\vec{b})$. $= 2\|\vec{a}\|^2 - \vec{a}\cdot\vec{b} - \|\vec{b}\|^2 = 2 - 0 - 9 = -7$ #

(c) (6 points) $\big((\vec{b}\cdot\vec{c})\vec{a} - (\vec{a}\cdot\vec{c})\vec{b}\big)\cdot\vec{c}$.

$= (-2\vec{a} - \vec{b})\cdot\vec{c} = -2\vec{a}\cdot\vec{c} - \vec{b}\cdot\vec{c}$

$= -2+2 = 0$ #

9. (6 points) Find the angle between $(1, 1, 0)$ and $(0, 1, 1)$.

$$= \cos^{-1}\left(\frac{(1,1,0)\cdot(0,1,1)}{\|(1,1,0)\|\cdot\|(0,1,1)\|}\right) = \cos^{-1}\frac{1}{2} = \frac{\pi}{3} \quad \#$$

題目分配：麟翔：1,2,7　　　　　登科：4,5,6　　　　　俊碩：3,8,9

- (2.c) 有同學計算兩個向量的內積後還是一個向量。

$(a_1, a_2, a_3) \bullet (b_1, b_2, b_3)$
$= a_1b_1 + a_2b_2 + a_3b_3 \in \mathbb{R}$

- (3.a) 同學們在做比較審斂的時候會寫下

$$\sum_{k=0}^{\infty}\left|\frac{(-1)^k}{k}-\frac{2}{k!}\right| \geq \sum_{k=0}^{\infty}\frac{1}{k}-\sum_{k=0}^{\infty}\frac{2}{k!}$$

?

$\Sigma(a_k+b_k) = \Sigma a_k + \Sigma b_k$
is ok only if both $\Sigma a_k$ and $\Sigma b_k$ converge

是不合邏輯的。

- (3.b) 這題無法使用交錯級數判別法。
- (5.c) 有同學用 root test 之後得到 $|x+2|<1$ 然後去檢查 $x=1,-1$ 這兩點，但是 $x=1$ 不是邊界點。
- (5.c) 當 $x=-3$ 時，我們要考慮的級數是 $\sum\limits_{k=1}^{\infty}(-1)^k\left(1+\frac{1}{k}\right)^k$。有同學證明此級數不滿足 alternating series test 的條件導致它會發散，但是 alternating series test 只能用來驗證此級數是否收斂，沒有給我們任何發散的資訊。
- (8.b) 同學們忘記了 $\vec{a}\cdot\vec{a}=\|\vec{a}\|^2$ ，並不是 $\|\vec{a}\|$

$(a_1, a_2, a_3) \bullet (a_1, a_2, a_3)$
$= a_1^2 + a_2^2 + a_3^2 = \|\vec{a}\|^2$

- (8.c) 有部分同學寫 $(\vec{b}\cdot\vec{c})\vec{a}=\vec{b}\cdot\vec{c}\cdot\vec{a}$，這是錯誤的。

$\vec{b}\bullet\vec{c} \in \mathbb{R}$