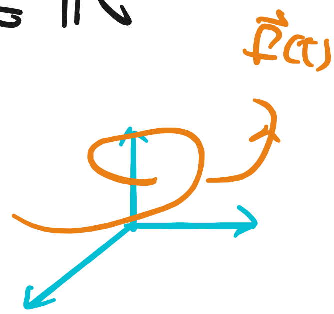
Calculus 4/23

Recall

Consider

$$\vec{f}(t) = (f_1(t), f_2(t), f_3(t)) \in \mathbb{R}^3$$

$$= f_1(t)\cdot\vec{i} + f_2(t)\cdot\vec{j} + f_3(t)\cdot\vec{k}$$



Then

$$\lim_{t\to c}\vec{f}(t) = \left(\lim_{t\to c} f_1(t),\ \lim_{t\to c} f_2(t),\ \lim_{t\to c} f_3(t)\right)$$

$$= \left(\lim_{t\to c} f_1(t)\right)\vec{i} + \left(\lim_{t\to c} f_2(t)\right)\vec{j} + \left(\lim_{t\to c} f_3(t)\right)\vec{k}$$

Def

A vector-valued function $\vec{f}$ is continuous at c if

$$\lim_{t\to c}\vec{f}(t) = \vec{f}(c)$$

Remark

$$\lim_{t\to c}\vec{f}(t) = \vec{f}(c) = (f_1(c), f_2(c), f_3(c))$$

$$\| \qquad\qquad //$$

$$\left(\lim_{t\to c} f_1(t), \lim_{t\to c} f_2(t), \lim_{t\to c} f_3(t)\right)$$

$$\Leftrightarrow \begin{cases} \lim_{t\to c} f_1(t) = f_1(c) \\ \lim_{t\to c} f_2(t) = f_2(c) \\ \lim_{t\to c} f_3(t) = f_3(c) \end{cases}$$

$\Leftrightarrow$ f_1, f_2 and f_3 are continuous at c

$\Leftrightarrow$ $\vec{f}$ is continuous at c

Def (Def. 14.1.5)

A vector-valued function $\vec{f}$ is differentiable at t if

$$\lim \frac{\vec{f}(t+h) - \vec{f}(t)}{} \quad \text{exists}$$

$\lim_{h\to 0}$ h

If this limit exists, it is called the derivative of $\vec{f}$ at t, denoted by

$\vec{f}'(t)$ or $\left.\frac{d\vec{f}}{dt}\right|_t$

Thm (page 697)

Let $\vec{f}(t) = (f_1(t), f_2(t), f_3(t))$ be a vector-valued function. Then $\vec{f}$ is differentiable at t iff all f_1, f_2 and f_3 are differentiable at t. In this case,

$$\boxed{\vec{f}'(t) = (f_1'(t), f_2'(t), f_3'(t))}$$

Pf

$$\vec{f}'(t) = \lim_{h\to 0} \frac{\vec{f}(t+h) - \vec{f}(t)}{h}$$

$$= \lim_{h\to 0} \cancel{\frac{1}{h}} \cdot \left(\frac{f_1(t+h) - f_1(t)}{h}, \frac{f_2(t+h) - f_2(t)}{h}, \frac{f_3(t+h) - f_3(t)}{h} \right)$$

$$= (\lim \frac{f_1(t+h) - f_1(t)}{} \quad \lim \frac{f_2(t+h) - f_2(t)}{}$$

$$= \left(\lim_{h\to 0} \frac{\quad}{h}, \lim_{h\to 0} \frac{\quad}{h}, \lim_{h\to 0} \frac{f_3(t+h)-f_3(t)}{h} \right)$$

$$= (f_1'(t), f_2'(t), f_3'(t)) \qquad \#$$

Example

① If $\vec{f}(t) = (1,2,3)$ is constant, then

$$\vec{f}'(t) = ((1)', (2)', (3)') = (0,0,0)$$

$$= \vec{0} \qquad \forall t.$$

② If $\vec{g}(t) = t\,\vec{i} + t^2\vec{j} - e^t\vec{k}$, then

$$\vec{g}'(t) = (t)'\,\vec{i} + (t^2)'\,\vec{j} + (-e^t)'\,\vec{k}$$

$$= \vec{i} + 2t\,\vec{j} - e^t\,\vec{k} \qquad \#$$

Remark

If $\vec{f}$ is differentiable at t, then

$\vec{f}$ is continuous at t.

<u>Thm</u> (§14.2)

Let $\vec{f}, \vec{g}$ be differentiable vector-valued functions

u be a differentiable real-valued function.

$\alpha, \beta \in \mathbb{R}$.

Then

(i) $(\alpha \vec{f} + \beta \vec{g})'(t) = \alpha \cdot \vec{f}'(t) + \beta \cdot \vec{g}'(t)$

(ii) $\underline{(u\vec{f})'(t)} = (u(t)\ \vec{f}(t))'$

$= (u(t) f_1(t),\ u(t) f_2(t),\ u(t) f_3(t))'$

$= \underline{u'(t)\ \vec{f}(t) + u(t)\ \vec{f}'(t)}$

(iii) $(\vec{f} \bullet \vec{g})'(t) = (\vec{f}(t) \bullet \vec{g}(t))'$

$= \underline{\vec{f}'(t) \bullet \vec{g}(t)} + \vec{f}(t) \bullet \vec{g}'(t)$

$(f_1(t) g_1(t) + f_2(t) g_2(t) + f_3(t) g_3(t))'$

$= (f_1(t) g_1(t))' + (f_2(t) g_2(t))' + (f_3(t) g_3(t))'$

$$= f_1'(t)\,g_1(t) + f_1(t)\,g_1'(t) + f_2'(t)\,g_2(t) + f_2(t)\,g_2'(t) + f_3'(t)\,g_3(t) + f_3(t)\,g_3'(t)$$

(iv) $$(\vec{f}\circ u)'(t) = \left(\vec{f}(u(t))\right)'$$

$$= u'(t)\,\vec{f}'(u(t))$$ — chain rule

$$\left(\left(f_1(u(t))\right)', \left(f_2(u(t))\right)', \left(f_3(u(t))\right)'\right)$$

$$= \left(f_1'(u(t))\cdot u'(t),\ f_2'(u(t))\cdot u'(t),\ f_3'(u(t))\cdot u'(t)\right)$$

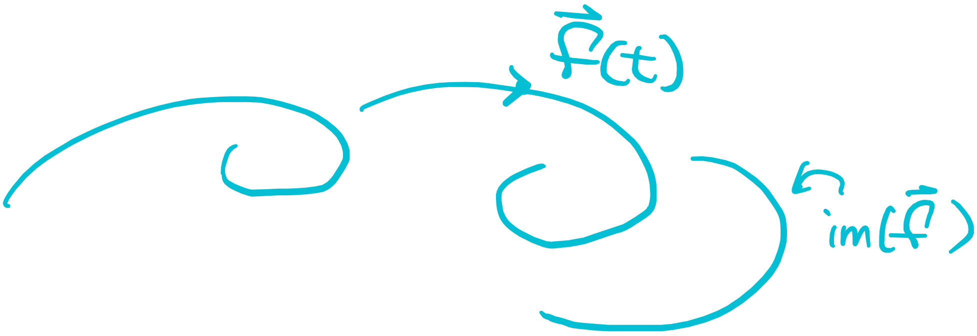
$$= u'(t)\,\vec{f}'(u(t))$$

Geometry of curve

A differentiable vector-valued function

$$\vec{f}(t) = f_1(t)\,\vec{i} + f_2(t)\,\vec{j} + f_3(t)\,\vec{k}$$

can be considered as a differentiable parametrized curve in $\mathbb{R}^3$.



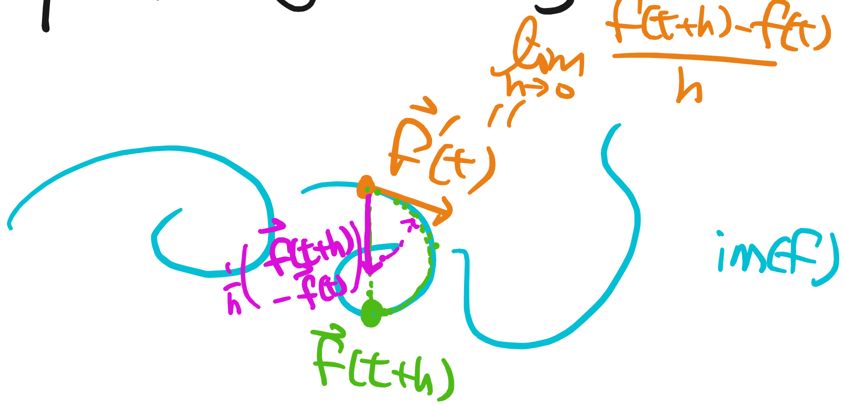
We consider $\vec{f}(t)$ as a parametrization of the curve $\mathrm{im}(\vec{f})$. For example,

$$\vec{f}(t) = \cos t\ \vec{i} + \sin t\ \vec{j}, \quad t \in \mathbb{R}$$

$$\vec{g}(t) = \cos 2\pi t\ \vec{i} + \sin 2\pi t\ \vec{j}, \quad t \in \mathbb{R}$$

are two parametrizations of the cirle

$$\{(x,y,0) \in \mathbb{R}^3 \mid x^2+y^2=1\}$$



__Def__

The vector

$$\vec{f}'(t) = f_1'(t)\ \vec{i} + f_2'(t)\ \vec{j} + f_3'(t)\ \vec{k}$$

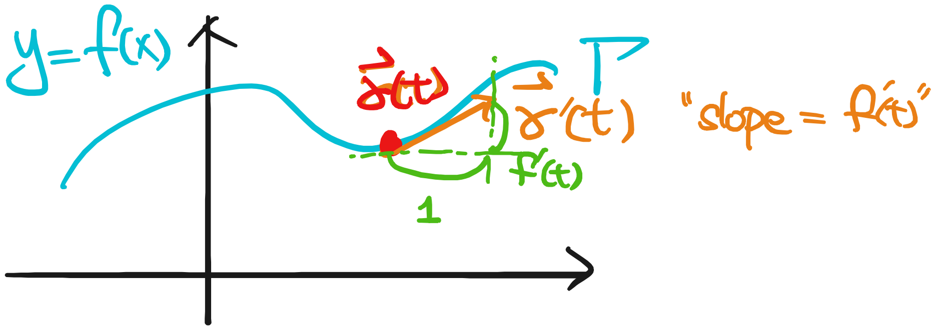
is a tangent vector of the curve $\mathrm{im}(\vec{f})$ at $\vec{f}(t)$.

切向量

__Remark__

Consider $f: \mathbb{R} \to \mathbb{R}$.

$$\Gamma = \{(x, f(x)) \in \mathbb{R}^2 \mid x \in \mathbb{R}\}$$



The vector-valued function

$$\vec{r}(t) = t\,\vec{i} + f(t)\,\vec{j}$$

is a parametrization of Γ

$$\Rightarrow \quad \vec{r}\,'(t) = 1\cdot\vec{i} + f'(t)\,\vec{j}$$

— a vector parallel to tangent line

HW5. 7

$$(1+x)^n = 1 + \frac{n!}{(n-1)!\,1!}x + \cdots + \frac{n!}{1!\,(n-1)!}x^{n-1} + x^n$$

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \cdots$$

(b) r = radius of convergence   Ratio Test

$$= \frac{1}{\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right|}$$

$$= \frac{1}{\lim_{k\to\infty} \left| \frac{\dfrac{\cancel{\alpha(\alpha-1)\cdots(\alpha-k+1)}(\alpha-k)}{\cancel{(k+1)!}}}{\dfrac{\cancel{\alpha(\alpha-1)\cdots(\alpha-k+1)}}{\cancel{k!}}} \right|} \qquad \alpha-k \quad k+1$$

$$= \frac{1}{\lim_{k\to\infty} \left| \frac{\frac{\alpha-k}{k}}{\frac{k+1}{k}} \right|} = \frac{1}{1} = 1$$

⇒ the series converges absolutely

on $(0-1, 0+1) = (-1, 1)$ #

(c) $$\varphi(x) = 1 + \sum_{k=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} x^k, \quad x\in(-1,1)$$

$$\Rightarrow \varphi'(x) = \sum_{k=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{(k-1)!} x^{k-1}, \quad x\in(-1,1)$$

$\Rightarrow (1+x)\varphi'(x)$ $\qquad$ $x^{k-1} + x^k$

$= \sum^{\infty} \alpha(\alpha-1)\cdots(\alpha-k+1)$

$$= \sum_{k=1} \frac{\qquad}{(k-1)!} \quad (1+x)\alpha^{k-1}$$

$$= \alpha + \frac{\alpha(\alpha-1)}{1!}x + \frac{\alpha(\alpha-1)(\alpha-2)}{2!}x^2 + \cdots \frac{\boxed{\alpha\cdots}(\alpha-k)}{k!}x^k$$

$$+ \quad \alpha \quad x + \frac{2\alpha(\alpha-1)}{2\times 1!}x^2 + \cdots \frac{k\boxed{\alpha\cdots(\alpha-k+1)}}{k(k-1)!}x^k$$

$$= \alpha + \alpha\cdot\alpha x + \frac{\alpha(\alpha-1)}{2!}\cdot\alpha x^2 + \cdots$$

$$= \alpha + \sum_{k=1}^{\infty} \frac{\boxed{\alpha\cdots(\alpha-k+1)}\cdot\alpha}{k!}x^k$$

$$\frac{\boxed{\alpha\cdots(\alpha-k+1)}(\alpha-k)}{k!}x^k$$

$$+\frac{k\boxed{\alpha\cdots(\alpha-k+1)}}{k(k-1)!}x^k$$

$$= \frac{\boxed{}(\alpha-\cancel{k}+\cancel{k})}{k!}x^k$$

$$= \alpha\cdot\varphi(x)$$

So

$$(1+x)\varphi'(x) = \alpha\varphi(x) \quad \forall x\in(-1,1)$$

$$\varphi(0) = 1$$

#

(d) Prove

$$\varphi(x) = (1+x)^\alpha \quad \forall x\in(-1,1)$$

Let

$$\psi(x) = \frac{\varphi(x)}{(1+x)^\alpha}, \quad x\in(-1,1)$$

Then

$$\psi'(x) = \frac{\varphi'(x)\cdot(1+x)^\alpha - \varphi(x)\cdot((1+x)^\alpha)'}{((1+x)^\alpha)^2}$$

$$((1+x)^\alpha)' = \alpha\cdot(1+x)^{\alpha-1}$$

$$= \underbrace{\left(\varphi'(x)\cdot(1+x) - \alpha\varphi(x)\right)}_{\overset{(c)}{=}\ 0} \cdot \frac{(1+x)^{\alpha-1}}{(1+x)^{2\alpha}}$$

$$= 0 \qquad \forall x \in (-1,1)$$

So $\psi(x) = $ a constant C

Since $\psi(0) = \dfrac{\varphi(0)}{(1+0)^{\alpha}} = \dfrac{1}{1} = 1$.

we have

$$\psi(x) \equiv 1 = \frac{\varphi(x)}{(1+x)^{\alpha}} \qquad \forall x \in (-1,1)$$

$$\Rightarrow \quad \varphi(x) = (1+x)^{\alpha} \qquad \forall x \in (-1,1) \qquad \#$$

演習課錯誤修正

俊碩

April 19, 2024

1 HW 4

- Problem 1.(f): 修正計算過程：

$$\sum_{k=1}^{\infty} \frac{\sin(\frac{\pi k}{2})}{k\sqrt{k}} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)\sqrt{2k+1}}$$

- Problem ~~3.(f)~~ 5(a): 修正答案：~~Interval of convergence is $|x-1| < \frac{3}{2}$，邊界 $x = \frac{5}{2}, \frac{-1}{2}$ 都不收斂。~~ 無邊界

2 HW 5

- Problem 1.(f): 修正過程與答案：

$$\cosh x \sinh x = \frac{1}{2}\sinh 2x = \frac{1}{2}\sum_{n=0}^{\infty} \frac{2^{2n+1}}{(2n+1)!}x^{2n+1}.$$

- Problem 2.(c): 修正答案

$$\frac{-9!}{7}$$

演習課時遺漏負號。

Summary of Taylor series and power series:

$$\boxed{\text{smooth functions } f(x) \underset{\text{Compute sum}}{\overset{\text{Taylor series}}{\rightleftarrows}} \text{power series } \sum_{k=0}^{\infty} a_k x^k}$$

- Taylor series: it starts with a function.
  - Taylor series of a smooth function f (at $x = 0$) is $\displaystyle\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$.
  - The series $\displaystyle\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$ does NOT necessarily converge to $f(x)$. For example, if
$$f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0, \end{cases}$$
then $f^{(n)}(0) = 0$ for all n. Thus, its Taylor series is the zero series which converges everywhere to zero, not $f(x)$.
  - To get actual $f(x)$, we have the finite expansion:
$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k + R_n(x),$$
where
$$R_n(x) = \frac{1}{n!} \int_0^x f^{(n+1)}(t) \cdot (x - t)^n dt = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}$$
for some c between x and 0.
  - The Taylor series $\displaystyle\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$ converges to $f(x)$ iff $R_n(x)$ converges to 0 as $n \to \infty$.
  - But $\displaystyle\lim_{n\to\infty} R_n(x)$ is difficult in general. Thus, we usually try to get the desired expansions by the power series techniques and the well-known expansions:
$$\begin{aligned} e^x &= \sum_{k=0}^{\infty} \frac{1}{k!} x^k = 1 + x + \frac{1}{2!} x^2 + \cdots, && x \in (-\infty, \infty); \\ \sin x &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 + \cdots, && x \in (-\infty, \infty); \\ \cos x &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} = 1 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4 + \cdots, && x \in (-\infty, \infty); \\ \frac{1}{1-x} &= \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots, && x \in (-1, 1); \\ \ln(1+x) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} x^{k+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots, && x \in (-1, 1]. \end{aligned}$$
  - Sometimes, we can get the expansion by differential equations or other techniques. For example, one can find an expansion of $(1 + x)^\alpha$ by the following steps: (i) compute the Taylor series of $(1 + x)^\alpha$; (ii) prove that the Taylor series converges to a function $\varphi(x)$ for $x \in (-1, 1)$; (iii) prove that $\varphi(x) = (1 + x)^\alpha$ via the differential equation
$$(1 + x)\varphi'(x) = \alpha\varphi(x).$$

- Power series: it start with a series $\sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + \cdots$. No function at beginning.
  - Many nice properties: radius of convergence, differentiation, integration, etc.
  - If $\sum_{k=0}^{\infty} a_k x^k$ converges on $(-r, r)$, $r > 0$, then we can consider the function

$$f(x) = \sum_{k=0}^{\infty} a_k x^k, \qquad x \in (-r, r).$$

    The Taylor series of $f(x)$ is exactly $\sum_{k=0}^{\infty} a_k x^k$.