Calculus 4/18

Recall

If $\vec{a} = (a_1, a_2, a_3)$, $\vec{b} = (b_1, b_2, b_3) \in \mathbb{R}^3$,

then

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

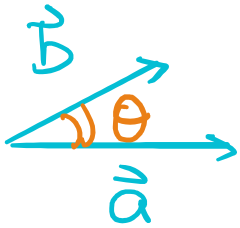
Thm

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \cdot \|\vec{b}\| \cdot \cos\theta$$

where

$\|\vec{a}\|$ = norm of $\vec{a}$ $= \sqrt{a_1^2 + a_2^2 + a_3^2}$

θ = angle between $\vec{a}$ and $\vec{b}$



Cor

$\vec{a} \perp \vec{b}$ iff $\vec{a} \cdot \vec{b} = 0$

垂直

$(\cos\theta = 0 \Leftrightarrow \theta = \pm\frac{\pi}{2})$

Example

Example

① The vectors

$(2,1,1)$ and $(1,1,-3)$

are perpendicular because
垂直

$$(2,1,1)\cdot(1,1,-3)$$
$$= 2\cdot 1 + 1\cdot 1 + 1\cdot(-3) = 0$$

② Assume the angle between

$(1,1,0)$ and $(0,1,1)$

is θ.

Then

$$\|(1,1,0)\|\,\|(0,1,1)\|\cos\theta$$
$$= (1,1,0)\cdot(0,1,1)$$
$$= 1\cdot 0 + 1\cdot 1 + 0\cdot 1 = 1$$

$$= \sqrt{1^2+1^2+0^2}\cdot\sqrt{0^2+1^2+1^2}\cdot\cos\theta$$

$$= \sqrt{2}\cdot\sqrt{2}\cdot\cos\theta = 2\cdot\cos\theta$$

$$\Rightarrow \quad \cos\theta = \frac{1}{2}$$

$$\Rightarrow \quad \theta = (\pm)\frac{\pi}{3} \qquad \#$$

<u>Prop</u>(§13.3)

Let $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^3$ and $\alpha, \beta \in \mathbb{R}$.

(i) $\vec{a} \cdot \vec{a} = \|\vec{a}\|^2 \quad (= a_1^2 + a_2^2 + a_3^2)$

(ii) $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

(iii) $\vec{a} \cdot (\alpha\vec{b} + \beta\vec{c}) = \alpha(\vec{a} \cdot \vec{b}) + \beta(\vec{a} \cdot \vec{c})$

$$\overset{//}{(\alpha\vec{b} + \beta\vec{c}) \cdot \vec{a}}$$

$$(a_1, a_2, a_3) \cdot (\alpha(b_1, b_2, b_3) + \beta(c_1, c_2, c_3))$$

$$= (a_1, a_2, a_3) \cdot (\alpha b_1 + \beta c_1, \alpha b_2 + \beta c_2, \alpha b_3 + \beta c_3)$$

$$= a_1(\alpha b_1 + \beta c_1) + a_2(\alpha b_2 + \beta c_2) + a_3(\alpha b_3 + \beta c_3)$$

$$= \alpha a_1 b_1 + \beta a_1 c_1 + \alpha a_2 b_2 + \beta a_2 c_2 + \alpha a_3 b_3 + \beta a_3 c_3$$

<u>Example</u>

Given that

$\|\vec{a}\| = 1 \quad \|\vec{b}\| = 3, \quad \|\vec{c}\| = 4$

$\|\vec{a}\|=1, \quad \|\vec{b}\| \ \ldots$

$\vec{a}\cdot\vec{b}=0, \quad \vec{a}\cdot\vec{c}=1, \quad \vec{b}\cdot\vec{c}=-2$

Compute the following:

① $3\vec{a}\cdot(\vec{b}+4\vec{c})$

$$= 3\vec{a}\cdot\vec{b} + (3\vec{a})\cdot(4\vec{c})$$

$$= 3\cdot 0 + 12\cdot \underbrace{\vec{a}\cdot\vec{c}}_{} {=1}$$

$$= 12$$

② $\left((\underbrace{\vec{b}\cdot\vec{c}}_{-2})\vec{a} - (\underbrace{\vec{a}\cdot\vec{c}}_{1})\vec{b}\right)\cdot\vec{c}$

$$= -2\,\underbrace{\vec{a}\cdot\vec{c}}_{1} - \underbrace{\vec{b}\cdot\vec{c}}_{-2}$$

$$= -2+2=0$$

③ $(\vec{a}-\vec{b})\cdot(2\vec{a}+\vec{b})$

$$= \vec{a}\cdot(2\vec{a}+\vec{b}) - \vec{b}\cdot(2\vec{a}+\vec{b})$$

$$= 2\underbrace{\vec{a}\cdot\vec{a}}_{\|\vec{a}\|^2=1} + \cancel{\vec{a}\cdot\vec{b}} \underbrace{\cancel{-2}}_{-1}\vec{b}\cdot\vec{a} - \underbrace{\vec{b}\cdot\vec{b}}_{\|\vec{b}\|^2=9}$$

$$= 2\cdot 1 - 1\cdot 0 - 9 = -7$$

#

<u>Remark</u>

All the definitions and properties about vectors can be established for vectors in $\mathbb{R}^2$, even in $\mathbb{R}^n$ for any n.

<u>Remark</u>

The authors use "i, j, k" notations:

$$i = \vec{i} = (1, 0, 0)$$
$$j = \vec{j} = (0, 1, 0)$$
$$k = \vec{k} = (0, 0, 1)$$

For example,

$$\vec{a} = (a_1, a_2, a_3)$$
$$= a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}$$

<u>Limit and vector derivative</u>

$\vec{r}: \mathbb{R} \to \mathbb{R}^3$

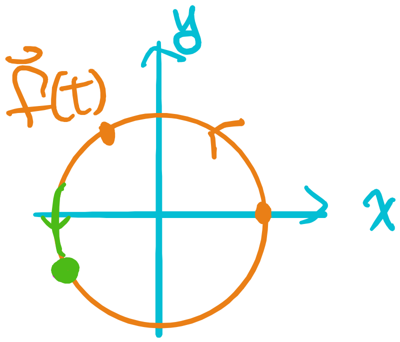
Consider $f : \mathbb{R} \to \mathbb{R}$

$$\vec{f}(t) = (f_1(t), f_2(t), f_3(t))$$
$$= f_1(t)\,\vec{i} + f_2(t)\,\vec{j} + f_3\,\vec{k}$$

" curves in $\mathbb{R}^3$ "

Example

① $\vec{f}(t) = (\cos t)\,\vec{i} + (\sin t)\,\vec{j}$



② $\vec{g}(t) = \cos(2\pi t)\,\vec{i} + \sin(2\pi t)\,\vec{j} + t\,\vec{k}$

Def (Def 14.1.1)

Let $\vec{f}(t)$ be a function valued in $\mathbb{R}^3$. We say the limit

$$\lim_{t \to t_0} \vec{f}(t)$$

$\lim_{t \to t_0} \vec{f}(t)$
$\parallel$

exists if there exists $\vec{L} \in \mathbb{R}^3$ s.t.

$$\lim_{t \to t_0} \| \vec{f}(t) - \vec{L} \| = 0$$

distance beteen $\vec{f}(t)$ and $\vec{L}$

<u>Thm</u> (14.1.4)

Let $\vec{f}(t) = (f_1(t), f_2(t), f_3(t))$

$\vec{L} = (L_1, L_2, L_3)$

Then

$$\lim_{t \to t_0} \vec{f}(t) = \vec{L}$$

iff

$$\begin{cases} \lim_{t \to t_0} f_1(t) = L_1 \\ \lim_{t \to t_0} f_2(t) = L_2 \\ \lim_{t \to t_0} f_3(t) = L_3 \end{cases}$$

<u>pf</u>

By the definition,

$$\lim_{t \to t_0} \vec{f}(t) = \vec{L}$$

$$\Leftrightarrow \quad \lim_{t \to t_0} \left\| \vec{f}(t) - \vec{L} \right\| = 0$$

$$\shortparallel$$

$$\lim_{t \to t_0} \sqrt{(f_1(t) - L_1)^2 + (f_2(t) - L_2)^2 + (f_3(t) - L_3)^2}$$

Since

$$0 \le |f_p(t) - L_p| \le \left\| \vec{f}(t) - \vec{L} \right\|, \quad \overrightarrow{} 0 \text{ as } t \to t_0$$

$p = 1, 2, 3$

by the pinching theorem,

$$\lim_{t \to t_0} |f_1(t) - L_1| = \lim_{t \to t_0} |f_2(t) - L_2|$$

$$= \lim_{t \to t_0} |f_3(t) - L_3| = 0$$

$$\Rightarrow \begin{cases} \lim_{t \to t_0} f_1(t) = L_1 \\ \lim_{t \to t_0} f_2(t) = L_2 \qquad (*) \\ \lim_{t \to t_0} f_3(t) = L_3 \end{cases}$$

$$\lim_{t\to t_0} f_3(t) = L_3$$

Conversely, assume ⊛ is true.

$$\Rightarrow \lim_{t\to t_0} \underbrace{\sqrt{(f_1(t)-L_1)^2+(f_2(t)-L_2)^2+(f_3(t)-L_3)^2}}_{\|\vec{f}(t)-\vec{L}\|} = 0$$

$$\Rightarrow \lim_{t\to t_0} \vec{f}(t) = \vec{L} \qquad \#$$

Example

Let

$$\vec{f}(t) = \cos(t+\pi)\,\vec{i} + \sin(t+\pi)\,\vec{j} + e^{-t^2}\,\vec{k}$$

Then

$$\lim_{t\to 0} \vec{f}(t)$$

$$= \Big(\underbrace{\lim_{t\to 0}\cos(t+\pi)}_{=\cos\pi = -1}\Big)\vec{i} + \Big(\underbrace{\lim_{t\to 0}\sin(t+\pi)}_{=\sin\pi = 0}\Big)\vec{j} + \Big(\underbrace{\lim_{t\to 0} e^{-t^2}}_{= e^{-0} = 1}\Big)\vec{k}$$

$$= -\vec{i} + \vec{k} = (-1, 0, 1) \qquad \#$$

Thm (Thm 14.1.3)

Let $\vec{f}$ and $\vec{g}$ be vector-valued functions and u a real-valued function

Suppose $\qquad \alpha, \beta \in \mathbb{R}$

$$\lim_{t\to t_0} \vec{f}(t) = \vec{L}, \quad \lim_{t\to t_0} \vec{g}(t) = \vec{M}$$

$$\lim_{t\to t_0} u(t) = A$$

Then

① $\displaystyle\lim_{t\to t_0} \left(\alpha \vec{f}(t) + \beta \vec{g}(t)\right) = \alpha \vec{L} + \beta \vec{M}$,

② $\displaystyle\lim_{t\to t_0} \left(u(t) \cdot \vec{f}(t)\right) = A\vec{L}$

③ $\displaystyle\lim_{t\to t_0} \|\vec{f}(t)\| = \|\vec{L}\|$

④ $\displaystyle\lim_{t\to t_0} \vec{f}(t) \bullet \vec{g}(t) = \vec{L} \bullet \vec{M}$

pf

④ $\lim \left(f_1(t), f_2(t), f_3(t)\right) \bullet \left(g_1(t), g_2(t), g_3(t)\right)$

$$\lim_{t\to t_0} \left(f_1(t), f_2(t), f_3(t)\right) \cdot \left(g_1(t), g_2(t), g_3(t)\right)$$

$$= \lim_{t\to t_0}\left(\underbrace{\overset{L_1}{f_1(t)}\,\overset{M_1}{g_1(t)}}_{L_1M_1} + \underbrace{\overset{L_2}{f_2(t)}\,\overset{M_2}{g_2(t)}}_{L_2M_2} + \underbrace{\overset{L_3}{f_3(t)}\,\overset{M_3}{g_3(t)}}_{L_3M_3} \right)$$

$$= L_1M_1 + L_2M_2 + L_3M_3 = \vec{L} \cdot \vec{M}$$

② $$\lim_{t\to t_0}\left(u(t)\,\vec{f}(t) \right)$$

$$= \lim_{t\to t_0} \left(u(t)\cdot f_1(t),\ u(t)\cdot f_2(t),\ u(t)\cdot f_3(t) \right)$$

$$= \left(\lim_{t\to t_0} \overset{A}{u(t)} \cdot \overset{L_1}{f_1(t)},\ \lim_{t\to t_0} \overset{A}{u(t)}\cdot \overset{L_2}{f_2(t)},\ \lim_{t\to t_0} \overset{A}{u(t)}\cdot \overset{L_3}{f_3(t)} \right)$$

$$= \left(AL_1,\ AL_2,\ AL_3 \right)$$

$$= A\cdot\left(L_1, L_2, L_3\right) = A\vec{L}$$

#

1 $(\cos t,\ \sin t) = \vec{f}(t)$

$0 \le t \le 10$

2 $(\cos 2\pi t,\ \sin 2\pi t,\ t) = \vec{g}(t)$

$-2 \le t \le 2$

3