Calculus 4/16

Vector calculus

So far we considered functions

$$f: \mathbb{R} \longrightarrow \mathbb{R}$$

— real-valued functions with one variable

From now on, we will consider

$$f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

In particular, we will consider

(i) $f: \mathbb{R} \longrightarrow \mathbb{R}^3$ (or $\mathbb{R}^2$)

向量

"curves in $\mathbb{R}^3$" or "vector-valued functions with one variable"

(ii) $f: \mathbb{R}^3$ (or $\mathbb{R}^2$) $\longrightarrow \mathbb{R}$

"multivariable real-valued function"

Vectors in $\mathbb{R}^3$

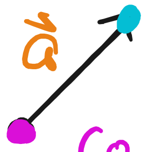
Vector = 向量 = direction + length

in $\mathbb{R}^3 = \{(x_1, x_2, x_3) \mid x_1, x_2, x_3 \in \mathbb{R}\}$

if same direction, same length

— uniquely determined by

a triple $\underbrace{(a_1, a_2, a_3)}_{=\vec{a}}$



<u>Def</u>

A vector $\vec{a}$ in $\mathbb{R}^3$ is an ordered triple of real numbers:

$$\vec{a} = (a_1, a_2, a_3) \in \mathbb{R}^3$$

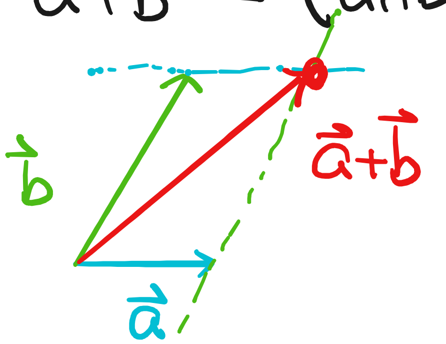
Let $\begin{matrix} \vec{a} = (a_1, a_2, a_3) \\ \vec{b} = (b_1, b_2, b_3) \end{matrix} \in \mathbb{R}^3$, $r \in \mathbb{R}$

(i) Equal:

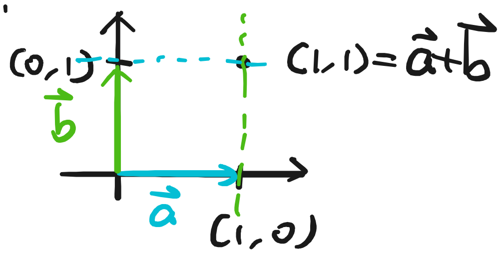
$$\vec{a} = \vec{b} \quad \text{iff} \quad \begin{cases} a_1 = b_1 \\ a_2 = b_2 \\ a_3 = b_3 \end{cases}$$

(ii) Addition:

$$\vec{a}+\vec{b} = (a_1+b_1,\ a_2+b_2,\ a_3+b_3)$$

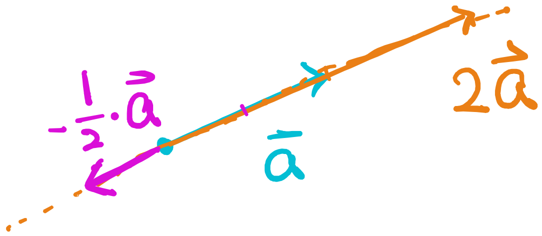


e.g.



(iii) <u>Scalar product:</u>

$$r.\vec{a} = (ra_1,\ ra_2,\ ra_3)$$



<u>Notations</u>

$$-\vec{a} = (-1)\cdot\vec{a}$$

$$\vec{a}-\vec{b} = \vec{a}+(-1)\cdot\vec{b}$$

$$\vec{0} = (0,0,0)$$

<u>Prop</u>

Let $\vec{a} = (a_1, a_2, a_3)$, $\vec{b} = (b_1, b_2, b_3) \in \mathbb{R}^3$
$r, s \in \mathbb{R}$ $\quad \vec{c} = (c_1, c_2, c_3)$

Then

(i) $\vec{a}+\vec{b} = \vec{b}+\vec{a}$

(ii) $(\vec{a}+\vec{b})+\vec{c} = \vec{a}+(\vec{b}+\vec{c})$

(iii) $\vec{a}+\vec{0} = \vec{a}$

(iv) $\vec{a}+(-\vec{a}) = \vec{0}$ "vector space"

(v) $1\cdot\vec{a} = \vec{a}$

(vi) $(r\cdot s)\cdot\vec{a} = r\cdot(s\cdot\vec{a})$

(vii) $r\cdot(\vec{a}+\vec{b}) = r\cdot\vec{a}+r\cdot\vec{b}$

(viii) $(r+s)\cdot\vec{a} = r\cdot\vec{a}+s\cdot\vec{a}$

pf $(r+s)\cdot(a_1,a_2,a_3) \overset{\text{def.}}{=} ((r+s)\cdot a_1,\ (r+s)\cdot a_2,\ (r+s)\cdot a_3)$

$= (ra_1+sa_1,\ ra_2+sa_2,\ ra_3+sa_3)$ 實數的分配律

$\overset{\text{def. of "+"}}{=} (ra_1,ra_2,ra_3)+(sa_1,sa_2,sa_3)$

$\overset{\text{def. "}\cdot\text{"}}{=} r(a_1,a_2,a_3)+s(a_1,a_2,a_3)$

$= r\cdot\vec{a}+s\cdot\vec{a}.$ #

Other properties can be proved similarly.

<u>Example</u>

Let $\vec{a}=(1,-1,2) \Rightarrow \|\vec{a}\| = \sqrt{1^2+(-1)^2+2^2} = \sqrt{6}$

Let $\vec{a} = (1,-1,2)$

$\vec{b} = (2,3,-1) \Rightarrow \|\vec{b}\| = \sqrt{2^2+3^2+(-1)^2} = \sqrt{14}$

$\vec{c} = (0,1,0) \Rightarrow \|\vec{c}\| = \sqrt{0^2+1^2+0^2} = 1$

① $\vec{a}-\vec{b} = \vec{a} + (-1)\cdot\vec{b}$

$= (1,-1,2) + (-1)\cdot(2,3,-1)$

$= (1,-1,2) + (-2,-3,1)$

$= (1-2,\ -1-3,\ 2+1) = (-1,-4,3)$ #

② $2\vec{a} + 3\vec{b} - \vec{c}$

$= 2\cdot(1,-1,2) + 3\cdot(2,3,-1) + (-1)\cdot(0,1,0)$

$= (2,-2,4) + (6,9,-3) + (0,-1,0)$

$= (2+6+0,\ -2+9+(-1),\ 4+(-3)+0)$

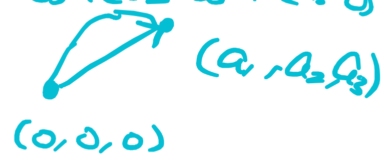
$= (8,\ 6,\ 1)$ #

Def

The norm (or length) of a vector

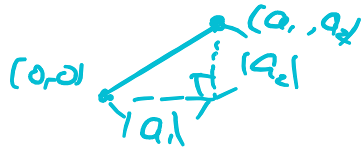
$\vec{a} = (a_1, a_2, a_3) \in \mathbb{R}^3$

$\sqrt{(a_1-0)^2+(a_2-0)^2+(a_3-0)^2}$



is the number

$$\|\vec{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$



We say $\vec{a}$ is a unit vector if 單位向量

$\|\vec{a}\| = 1$

Prop.

Let $\vec{a}, \vec{b} \in \mathbb{R}^3$, $r \in \mathbb{R}$. Then

(i) $\|\vec{a}\| \geq 0$, and

$\|\vec{a}\| = 0$ iff $\vec{a} = \vec{0}$

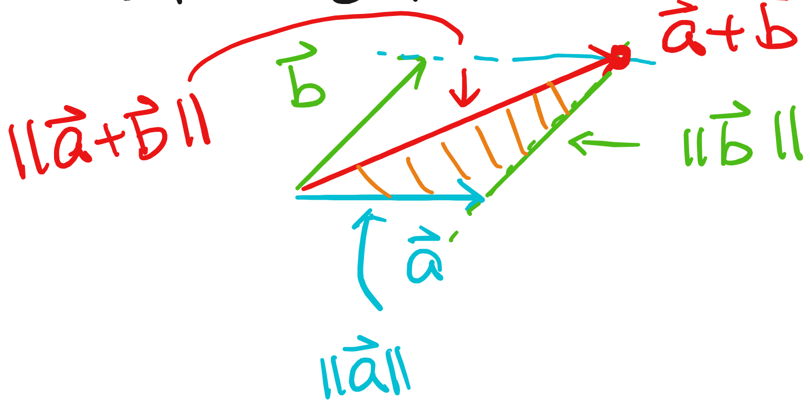
(ii) $\|r \cdot \vec{a}\| = |r| \cdot \|\vec{a}\|$

$$\| (ra_1, ra_2, ra_3) \| = \sqrt{r^2 a_1^2 + r^2 a_2^2 + r^2 a_3^2}$$

$$= \sqrt{r^2} \cdot \sqrt{a_1^2 + a_2^2 + a_3^2}$$

$$= |r| \cdot \|\vec{a}\|$$

(iii) $\|\vec{a} + \vec{b}\| \leq \|\vec{a}\| + \|\vec{b}\|$ triangular inequality 三角不等式



Remark

1-dimensional case:

$$\vec{a} = a \in \mathbb{R}^1$$

$$\|\vec{a}\| = \sqrt{a^2} = |a|$$

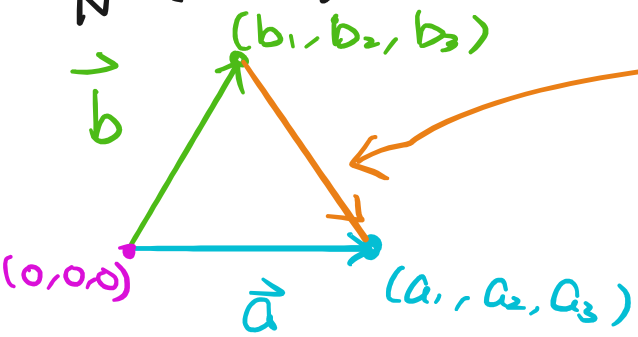
$$\text{(iii)} \Rightarrow \quad |a+b| \leq |a| + |b|$$

Remark

$$\|\vec{a} - \vec{b}\| = \|(a_1 - b_1,\ a_2 - b_2,\ a_3 - b_3)\|$$

= 距離

$$= \sqrt{(a_1-b_1)^2 + (a_2-b_2)^2 + (a_3-b_3)^2}$$



= distance between (a_1, a_2, a_3) and (b_1, b_2, b_3)

Example

$\vec{a} = (1, -1, 2)$, $\vec{b} = (2, 3, -1)$

① $\|\vec{a}\| = \sqrt{6}$, $\|\vec{b}\| = \sqrt{14}$

② $\|\vec{a}+\vec{b}\| = \sqrt{(1+2, -1+3, 2+(-1))}$ $\|\vec{a}\| + \|\vec{b}\|$

$= \sqrt{3^2+2^2+1^2} = \sqrt{14} < \sqrt{6}+\sqrt{14}$

③ The distance between $(1,-1,2)$ and $(2,3,-1)$

$= \|\vec{a}-\vec{b}\| = \|(1-2, -1-3, 2-(-1))\|$

$= \sqrt{(-1)^2+(-4)^2+3^2} = \sqrt{26}$ #

Def (Def 13.3.1)

Let $\vec{a} = (a_1, a_2, a_3)$, $\vec{b} = (b_1, b_2, b_3) \in \mathbb{R}^3$.

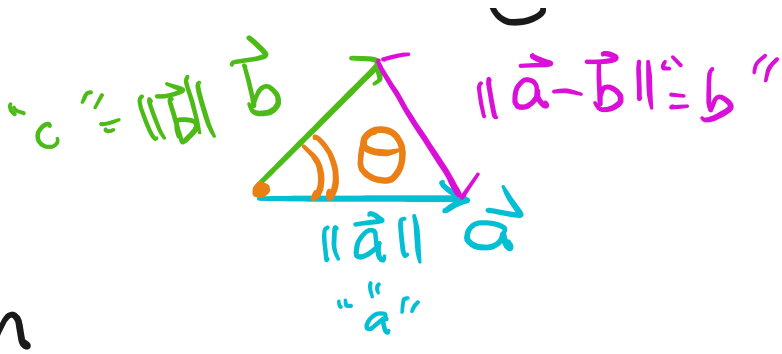
The dot product of $\vec{a}$ and $\vec{b}$ is

$$\vec{a} \bullet \vec{b} = a_1b_1 + a_2b_2 + a_3b_3 \in \mathbb{R}$$

The geometric meaning of $\vec{a}\cdot\vec{b}$ is

Thm

Let θ be the angle from $\vec{a}$ to $\vec{b}$

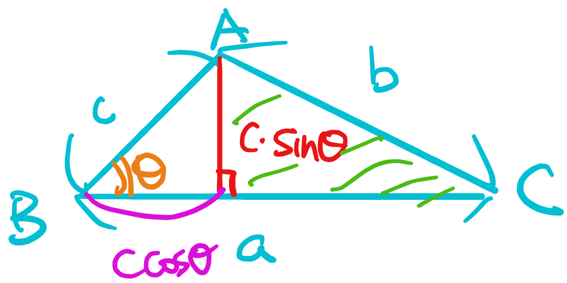


Then

$$\vec{a}\bullet\vec{b} = \|\vec{a}\|\cdot\|\vec{b}\|\cdot\cos\theta$$

pf

<u>Recall</u>: Law of cosine



$$\boxed{b^2} = (c\sin\theta)^2 + (a - c\cos\theta)^2$$

$$= \underline{c^2\sin^2\theta} + a^2 - 2ac\cos\theta + \underline{c^2\cos^2\theta}$$

$$= \boxed{a^2 - 2ac\cos\theta + c^2}$$

By this formula,

$$\|\vec{a}-\vec{b}\|^2 = \|\vec{a}\|^2 + \|\vec{b}\|^2 - 2\|\vec{a}\|\|\vec{b}\|\cos\theta$$

$$\Rightarrow \|\vec{a}\|\cdot\|\vec{b}\|\cdot\cos\theta = \frac{1}{2}\left(\|\vec{a}\|^2 + \|\vec{b}\|^2 - \|\vec{a}-\vec{b}\|^2\right)$$

$$= \frac{1}{2}\left((\cancel{a_1^2} + \cancel{a_2^2} + \cancel{a_3^2}) + (\cancel{b_1^2} + \cancel{b_2^2} + \cancel{b_3^2}) - \left(\underbrace{(a_1-b_1)^2 + (a_2-b_2)^2 + (a_3-b_3)^2}_{\cancel{a_1^2} + \cancel{b_1^2} - 2a_1b_1 + \cancel{a_2^2} + \cancel{b_2^2} - 2a_2b_2 + \cancel{a_3^2} + \cancel{b_3^2} - 2a_3b_3}\right) \right)$$

$$= a_1b_1 + a_2b_2 + a_3b_3 = \vec{a} \cdot \vec{b}$$

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