Calculus 4/11

Example

Find the Taylor series of $\tan^{-1} x$.

sol

Recall

- $$(\tan^{-1} x)' = \frac{1}{1+x^2}$$

- $$\frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{k=0}^{\infty} x^k, \quad \forall x \in (-1,1)$$

So, for $x \in (-1,1)$, $\boxed{-x^2} \in \cancel{(-1,1)}\ [-1,0] \subset (-1,1)$

$$\Rightarrow \quad \frac{1}{1-(-x^2)} = \frac{1}{1+x^2} \qquad \forall x \in (-1,1)$$

$$\frac{1}{1-(-x^2)} = \sum_{k=0}^{\infty} (-x^2)^k = \sum_{k=0}^{\infty} (-1)^k x^{2k}$$

Therefore,

$$\tan^{-1} x = \int \sum_{k=0}^{\infty} (-1)^k x^{2k} \, dx + C$$

$$= \sum^{\infty} (-1)^k \int x^{2k} \, dx + C$$

$$k=0$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1} + C$$

At $x=0$:

$$\tan^{-1} 0 = 0 = \underbrace{\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} 0^{2k+1}}_{=0} + C = C$$

$$\Rightarrow \quad C = 0$$

<u>Conclusion</u>:

$$\tan^{-1} x = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}, \quad \forall x \in (-1, 1)$$

#

<u>Summary of the method for finding Taylor series</u>:

<u>Method I</u>

(a) Compute $f^{(n)}(0)$ $\forall n$.

$$\Rightarrow \quad f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} \cdot x^k + R_n(x)$$

(b) Study for what x,

$$\boxed{\lim_{n\to\infty} R_n(x) = 0} \Leftrightarrow f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$$

<u>Method II</u>

Compare $f(x)$ with those known expansions:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}$$

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k, \quad x \in (-1, 1)$$

Use differentiation, integration, and other operations

National Tsing Hua University

Calculus II – Exam 1

Instructor: Hsuan-Yi Liao

Fall, 2023

Name: Answer

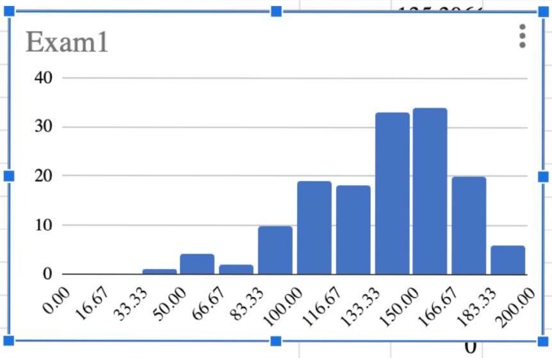
Student ID: ______________________

- This exam contains 8 pages (including this cover page) and 8 questions.
- Total of points is 200.
- Time limit: **100 minutes**.
- Write down your computation or arguments in details unless otherwise stated.
- The use of a calculator, cell phone, or any other electronic device is **NOT** permitted.
- The use of books or notes of any kind is **NOT** permitted.
- The use of L'Hôspital's rule is **NOT** allowed in this exam.

Distribution of Marks

Question	Points	Score
8	24	
Total:	200	

Average	138.0680
Average (except zero)	138.0680
Quartile 0	36
Quartile 1	117
Quartile 2	142
Quartile 3	162
Quartile 4	196
標準差	31.17112
非零數	147
不到60%人數	39
滿分人數	0



1. State whether the sequence converges. If it does, find the limit. If it doesn't, explain why.

   (a) (8 points) $a_n = 2^n$. unbounded $\Rightarrow$ diverges

   (b) (8 points) $a_n = \dfrac{2^n}{4^n+1}$. $\to 0$

   (c) (8 points) $\boxed{a_n = \dfrac{(n+1)(n+2)}{(n+3)(n+4)}.}$ $\to 1$

   (d) (8 points) $a_n = \dfrac{\ln(n+1)}{n}$. $\to 0$

   (e) (8 points) $a_n = \dfrac{4^{100n}}{n!}$. $\to 0$

   (f) (8 points) $a_n = n^2 \sin\dfrac{\pi}{n}$. diverges

$$n\pi \underbrace{\frac{\sin\frac{\pi}{n}}{\frac{\pi}{n}}}_{\to 1}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(n+1)(n+2)/n^2}{(n+3)(n+4)/n^2}$$

$$= \lim_{n\to\infty} \frac{\overbrace{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}^{\to 1}}{\underbrace{\left(1+\frac{3}{n}\right)\left(1+\frac{4}{n}\right)}_{\to 1}}$$ as $n\to\infty$

$$= \frac{1}{1} = 1.$$

(g) (8 points) $a_n = \int_0^{1/n} \tan(e^{\sin x})\, dx$. $\to 0$

(h) (8 points) $a_n = \left(\frac{1}{2} + \frac{3}{n}\right)^{3n}$. $\to 0$

2. Prove the following.

   (a) (8 points) $\ln(\ln x) = o(\ln x)$.

   (b) (8 points) $3x^5 - 100x^2 + 5x + 1 = O(x^5)$.

(a) $\lim_{x\to\infty} \frac{\ln(\ln x)}{\ln x} = \lim_{x\to\infty} \frac{\frac{1}{\ln x}\cdot\cancel{\frac{1}{x}}}{\cancel{\frac{1}{x}}} = 0 \Rightarrow \ln(\ln x) = o(\ln x)$ #

(b) Since $\lim_{x\to\infty} \frac{3x^5 - 100x^2 + 5x + 1}{x^5} = 3$, $\exists M$ s.t. $\left|\frac{3x^5-100x^2+5x+1}{x^5} - 3\right| < \varepsilon = 1$

For $\varepsilon = 1$, $\exists M$ s.t. $\forall x \geq M$

$\Downarrow$

$-1 < \frac{3x^5-100x^2+5x+1}{x^5} - 3 < 1$

$2 < \frac{3x^5 - 100x^2 + 5x + 1}{x^5} \underline{\underline{\leq 4}}$ $\forall x \geq M$.

$\Rightarrow 3x^5 - 100x^2 + 5x + 1 = O(x^5)$ #

3. Does the integral converge or diverge? Explain your answer.

(a) (8 points) $\int_0^{\pi/2} \tan x\, dx.$ $= \lim_{b \to \frac{\pi}{2}^-} \int_0^b \frac{\sin x}{\cos x} dx = \lim_{b \to \frac{\pi}{2}^-} \left(-\ln|\cos x|\right)\Big|_0^b$

$= \lim_{b \to \frac{\pi}{2}} (-\ln \cos b)$ diverges #

(b) (8 points) $\int_1^{\infty} \frac{\sqrt{x+1}}{x^2} dx.$

(c) (8 points) $\int_{\pi}^{\infty} \frac{2+\cos x}{x} dx.$

(b) $\lim_{x \to \infty} \frac{\frac{\sqrt{x+1}}{x^2}}{x^{-\frac{3}{2}}} = 1$ and $\int_1^{\infty} x^{-\frac{3}{2}} dx$ converges

$\Rightarrow \int_1^{\infty} \frac{\sqrt{x+1}}{x^2} dx$ converges #

(c) $\frac{2+\cos x}{x} \geq \frac{1}{x} > 0 \quad \forall x \geq \pi$

and $\int_{\pi}^{\infty} \frac{1}{x} dx$ diverges

$\Rightarrow \int_{\pi}^{\infty} \frac{2+\cos x}{x} dx$ diverges. #

4. Evaluate the integrals.

(a) (12 points) $\displaystyle\int_{-\infty}^{\infty} \frac{1}{e^x + e^{-x}}\, dx.$ $= \displaystyle\int_0^{\infty} \frac{1}{e^x+e^{-x}}dx + \int_{-\infty}^{0} \frac{1}{e^x+e^{-x}}dx$

(b) (12 points) $\displaystyle\int_0^1 x \ln x\, dx.$

$u = e^x \Rightarrow du = e^x dx$

$$\int_0^b \frac{1}{e^x+e^{-x}}dx = \int_0^b \frac{e^x}{e^{2x}+1}dx$$

$$= \int_1^{e^b} \frac{1}{u^2+1}du = \tan^{-1}e^b - \tan^{-1}1$$

$$\longrightarrow \frac{\pi}{2} - \frac{\pi}{4} \text{ as } b \to \infty$$

$$\int_a^0 \frac{1}{e^x+e^{-x}}dx = \int_{e^a}^1 \frac{1}{u^2+1}du$$

$$= \frac{\pi}{4} - \tan^{-1}e^a \longrightarrow \frac{\pi}{4} \text{ as } a \to -\infty$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{1}{e^x+e^{-x}}dx = \frac{\pi}{2} \ \#$$

b) $\displaystyle\int_b^1 x\ln x\, dx = \frac{x^2}{2}\ln x\Big|_b^1 - \int_b^1 \frac{x^{\cancel{2}}}{2}\cdot\frac{1}{\cancel{x}}dx = -\frac{b^2\cdot\ln b}{2} - \frac{x^2}{4}\Big|_b^1$

$$= -\frac{b^2\ln b}{2} - \frac{1}{4} + \frac{b}{4} \longrightarrow -\frac{1}{4} \quad \text{as } b \to 0^+ \ \#$$

5. Let j be a positive integer.

   (a) (8 points) Show that

$$\sum_{k=0}^{\infty} a_k \quad \text{converges} \qquad \text{iff} \qquad \sum_{k=j}^{\infty} a_k \quad \text{converges.}$$

   (b) (4 points) Show that if $\sum_{k=0}^{\infty} a_k = L$, then $\sum_{k=j}^{\infty} a_k = L - \sum_{k=0}^{j-1} a_k$.

   (c) (4 points) Show that if $\sum_{k=j}^{\infty} a_k = M$, then $\sum_{k=0}^{\infty} a_k = M + \sum_{k=0}^{j-1} a_k$.

(i) Assume $\sum_{k=0}^{\infty} a_k$ Converges to L.

$\Rightarrow \forall \varepsilon > 0 \ \exists N$ s.t. $\left| \sum_{k=0}^{n} a_k - L \right| < \varepsilon \quad \forall n \geq N$

$$\Rightarrow \left| \sum_{k=j}^{n} a_k - \left(L - \sum_{k=0}^{j-1} a_k \right) \right| = \left| \sum_{k=0}^{n} a_k - L \right| < \varepsilon \quad \forall n \geq N$$

$\Rightarrow \sum_{k=j}^{\infty} a_k$ Converges to $L - \sum_{k=0}^{j-1} a_k$

(ii) Assume $\sum_{k=j}^{\infty} a_k$ Converges to M, i.e. $\forall \varepsilon > 0 \ \exists K$ s.t.

$$\left| \sum_{k=j}^{n} a_k - M \right| < \varepsilon \quad \forall n \geq K$$

$$\Rightarrow \left| \sum_{k=0}^{n} a_k - \left(M + \sum_{k=0}^{j-1} a_k \right) \right| = \left| \sum_{k=j}^{n} a_k - M \right| < \varepsilon \quad \forall n \geq K$$

$\Rightarrow \sum_{k=0}^{\infty} a_k$ Converges to $M + \sum_{k=0}^{j-1} a_k$.

(i) + (ii) $\Rightarrow$ (a), (b), (c) $\blacksquare$

6. Determine whether the series converges or diverges. No need to explain.

C (a) (2 points) $\sum_{k=2}^{\infty} \frac{k}{k^3 - k}$.

$\left(\frac{\ln x}{x}\right)' = \frac{\frac{1}{x}\cdot x - \ln x \cdot 1}{x^2} = \frac{1-\ln x}{x^2} < 0$ for $x \geq 3$

C (b) (2 points) $\sum_{k=1}^{\infty} (-1)^k \frac{\ln k}{k}$.

C (c) (2 points) $\sum_{k=1}^{\infty} k^2 2^{-k^3}$. $\sqrt[k]{k^2 2^{-k^3}} = (\sqrt[k]{k})^2 \cdot 2^{-k^2} \to 0 < 1$

D (d) (2 points) $\sum_{k=1}^{\infty} \frac{2+\cos k}{\sqrt{k+1}}$. $\frac{2+\cos k}{\sqrt{k+1}} \geq \frac{1}{\sqrt{k+1}}$

D (e) (2 points) $\sum_{k=1}^{\infty} (-1)^k \frac{e^k}{k^2}$. $\to \infty$ as $k \to \infty$

C (f) (2 points) $\sum_{k=1}^{\infty} \frac{k^k}{3^{k^2}}$. $\sqrt[k]{\frac{k^k}{3^{k^2}}} = \frac{k}{3^k} \to 0 < 1$

C (g) (2 points) $\sum_{k=1}^{\infty} \frac{2\cdot 4 \cdots 2k}{(2k)!}$. $\frac{2\cdot 4 \cdots 2(k+1)}{(2(k+1))!} \Big/ \frac{2\cdots 2k}{(2k)!} = \frac{2(k+1)}{(2k+1)(2k+2)} \to 0 < 1$

D (h) (2 points) $\sum_{k=1}^{\infty} \frac{k!}{k^{k/2}}$. $\to \infty$ as $k \to \infty$

7. Is the series absolutely convergent? Explain your answer.

(a) (8 points) $\sum_{k=1}^{\infty} \left(\frac{(-1)^k}{k} - \frac{2}{k!}\right)$.

(b) (8 points) $\sum_{k=1}^{\infty} \frac{(-1)^k}{\ln(1+k)}$.

(a) $\left|\frac{(-1)^k}{k} - \frac{2}{k!}\right| \geq \frac{1}{k} - \frac{2}{k!} \geq 0 \ \forall k \geq 4$

Since $\sum \frac{1}{k}$ diverges and $\sum \frac{2}{k!}$ Converges,

we have $\sum \frac{1}{k} - \frac{2}{k!}$ diverges

Comparison $\Rightarrow \sum \left|\frac{(-1)^k}{k} - \frac{2}{k!}\right|$ diverges

$\Rightarrow$ NOT absolutely convergent. #

Note: $\sum_{k=0}^{\infty} k$, $\sum_{k=0}^{\infty} (-k)$ diverge

$\sum_{k=0}^{\infty} k + (-k)$ converges

b) Let $f(x) = \ln(1+x) - x$

$f(0) = 0$,

$f'(x) = \frac{1}{1+x} - 1 \leq 0 \ \forall x \geq 0$

$\Rightarrow f(x) \leq 0 \ \forall x \geq 0$

$\Rightarrow \ln(1+x) \leq x \ \forall x \geq 0$

$\Rightarrow \left|\frac{(-1)^k}{\ln(1+k)}\right| = \frac{1}{\ln(1+k)} \geq \frac{1}{k} \ \forall k \geq 1$

Since $\sum \frac{1}{k}$ diverges, we have $\sum \frac{1}{\ln(1+k)}$ diverges

$\Rightarrow$ NOT absolutely convergent. #

8. (24 points) Prove that

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

for any real number x.

By Taylor's Thm, $\exists$ c between 0 and x s.t.

$$e^x = \sum_{k=0}^{n} \frac{x^k}{k!} + \frac{e^c}{(n+1)!} x^{n+1}$$

$$\Rightarrow \left| e^x - \sum_{k=0}^{n} \frac{x^k}{k!} \right| = e^c \cdot \frac{|x|^{n+1}}{(n+1)!} \leq e^{|x|} \cdot \frac{|x|^{n+1}}{(n+1)!} \to 0$$

as $n \to \infty$

$$\Rightarrow \sum_{k=0}^{\infty} \frac{x^k}{k!} = \lim_{n \to \infty} \sum_{k=0}^{n} \frac{x^k}{k!} = e^x \quad \#$$

1 登科

- (1.g) 有一些同學寫 $\lim_{n\to\infty}(\int_0^{1/n}\tan(e^{\sin x})dx) = \int_0^0 \tan(e^{\sin x})dx = 0$ 但沒有解釋積分裡面的函數是否連續。
- (1.h) 不少同學寫 $\lim_{n\to\infty}(\frac{1}{2}+\frac{3}{n})^{3n} = \lim_{n\to\infty}(\frac{1}{2}+0)^{3n} = 0$。
- (2b) 大部分的人只有證明"f and g grow at the same rate" (i.e. $\lim_{x\to\infty}\frac{3x^5-100x^2+5x+1}{x^5} = 3$) 然後沒有解釋為什麼會存在一個 M 使得當 x 夠大的時候，$\frac{3x^5-100x^2+5x+1}{x^5} < M$ 就結論 $3x^5-100x^2+5x+1 = O(x^5)$。

2 麟翔

$\lim_{b\to\frac{\pi}{2}} \int_0^b \tan x \, dx$

- (3) 跟 (4) 很多人瑕積分會直接進行計算，而沒有寫出極限過程。例如

積?

$$\int_0 \tan x dx = \ln|\sec x|_0^{2\pi} = \ln x|_1^0$$

$\frac{\sin x}{\cos x}$

diverge because $\ln 0$ does not exist.

- (4.a) 很多人會寫

$$\int_{-\infty}^{\infty}\frac{1}{e^x-e^{-x}}dx = \lim_{b\to\infty}\int_{-b}^{b}\frac{1}{e^x-e^{-x}}dx$$

- (5) 幾乎所有人都寫 ” 因為有等式

$$\sum_{k=1}^{\infty}a_k = \sum_{k=1}^{j}a_k + \sum_{k=j+1}^{\infty}a_k$$

所以

$$\sum_{k=1}^{\infty}a_k \text{ 收斂} \iff \sum_{k=j+1}^{\infty}a_k \text{ 收斂。”}$$

但我認為直接寫上述等式，本題等於沒有證明。 can be proved by the definition of limit

3 俊碩

- (7.a) 大部分學生在絕對值計算上有誤：

$\frac{1}{k}-\frac{2}{k!} = \frac{1}{1}-\frac{2}{1!} = 1-2 = -1$ If $k=1$, $\left|\frac{(-1)^k}{k}-\frac{2}{k!}\right| = \left|\frac{(-1)^1}{1}-\frac{2}{1!}\right|$

$$\left|\frac{(-1)^k}{k}-\frac{2}{k!}\right| \geq \frac{1}{k}-\frac{2}{k!}$$

$= |1-2| = |-1| = 1$

也沒有說明 $\sum\frac{2}{k!}$ 收斂，直接由 $\sum 1/k$ 發散得出結論。此外，同學們似乎不熟悉三角不等式。
- (8) 大部分同學有寫出 Taylor 展開，但沒有寫下餘項或寫下餘項沒有驗證會隨 n 變大而趨近 0。

HW2

5. This problem shows that $\int_{-\infty}^{\infty} f(x)\,dx$ and $\lim_{b\to\infty}\int_{-b}^{b} f(x)\,dx$ are different.

(a) Show that $\int_0^{\infty}\frac{2x}{x^2+1}\,dx$ diverges and hence that $\int_{-\infty}^{\infty}\frac{2x}{x^2+1}\,dx$ diverges.

(b) Show that

$$\lim_{b\to\infty}\int_{-b}^{b}\frac{2x}{x^2+1}\,dx = 0.$$

i.g.

$$\lim_{n\to\infty} \int_0^{\frac{1}{n}} \tan(e^{\sin x})\,dx = ?$$

$$\neq \int_0^0 \tan(e^{\sin x})\,dx$$

$$= \lim_{n\to\infty} F(\tfrac{1}{n}) \overset{\text{hope}}{=} F(\lim_{n\to\infty} \tfrac{1}{n})$$

it is true if $F(y)$ is continuous at $y=0$

sol

Let

$$F(y) = \int_0^y \tan(e^{\sin x})\,dx$$

Recall (Fundamental Thm of Calculus)

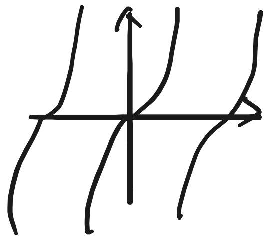
If $f(x)$ is $\boxed{\text{continuous}}$ on $[a,b]$, then

$$F(y) = \int_{\cancel{a}\,c}^y f(x)\,dx \quad , \quad c\in(a,b)$$

is differentiable $\forall\, y\in(a,b)$, and

$\Rightarrow$ continuous

$$F'(y) = f(y) \quad \forall\, y\in(a,b).$$



Q: Is $\tan(\boxed{e^{\sin x}})$ continuous?

θ

Note: $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ is NOT continuous at $\theta = \pm\frac{\pi}{2} + 2k\pi$

Q: Is $e^{\sin x} = \pm\frac{\pi}{2} + 2k\pi$?

Note:

$$F(y) = \int_0^y \tan(e^{\sin x})\,dx$$

We only care $y=0$!! That is, when x is close to 0.

(y is near 0)

Note:

At $x=0$,

$$e^{\sin 0} = e^{0} = 1 \neq \pm\frac{\pi}{2} + 2k\pi$$

Since $e^{\sin x}$ is continuous,

when x is near $x=0$,

$$e^{\sin x} \neq \pm\frac{\pi}{2} + 2k\pi$$

$\Rightarrow$ $\tan(e^{\sin x})$ is continuous when x is near 0.

$\Rightarrow$ $F(y) = \int_0^y \tan(e^{\sin x})\, dx$ is continuous at $y=0$

$$\Rightarrow \lim_{n\to\infty} F\left(\frac{1}{n}\right) = F\left(\lim_{n\to\infty} \frac{1}{n}\right) = F(0) = \int_0^0 \tan(e^{\sin x})\, dx = 0$$

$$\lim_{n\to\infty} F\left(\frac{1}{n}\right) = \lim_{n\to\infty} \int_0^{\frac{1}{n}} \tan(e^{\sin x})\, dx$$

#

(1.h)

$$\cancel{\lim_{n\to\infty}\left(\frac{1}{2}+\frac{3}{n}\right)^{\boxed{3n}} = \lim_{n\to\infty}\left(\lim_{n\to\infty}\left(\frac{1}{2}+\frac{3}{n}\right)\right)^{3n} = \lim_{n\to\infty}\left(\frac{1}{2}+0\right)^{3n} = 0}$$

和 n 有關

和 n 無關

$$\lim_{n\to\infty}\left(\frac{1}{2}+\frac{3}{n}\right)^{\boxed{5}} = \left(\lim_{n\to\infty}\left(\frac{1}{2}+\frac{3}{n}\right)\right)^{5} \quad \text{ok}$$

NOTE:

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$

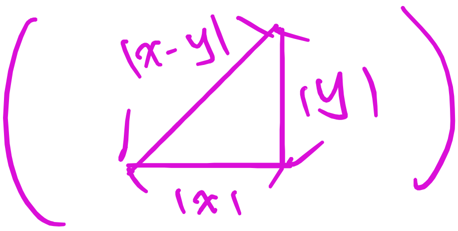
$$\lim_{n\to\infty}\left(\lim_{n\to\infty}\left(1+\frac{1}{n}\right)\right)^n = \lim_{n\to\infty}(1)^n = 1$$

三角不等式：

(i) $|a+b| \le |a|+|b|$

$$|a+b| \le |a|+|b|$$

$$|\underbrace{a+b}_{x}| - |\underbrace{b}_{y}| \le |\underbrace{a}_{x-y}|$$



(ii) $|x-y| \ge |x|-|y|$

$$\Downarrow$$

$$\left|\frac{(-1)^k}{k} - \frac{2}{k!}\right| \ge \left|\frac{(-1)^k}{k}\right| - \left|\frac{2}{k!}\right|$$

$$= \frac{1}{k} - \frac{2}{k!}$$