Calculus 4/9

Consider differentiation and integration of

$$\sum_{k=0}^{\infty} a_k x^k$$

<u>Thm</u> (Thm 12.9.1 & 12.9.2)

If

$$f(x) = \sum_{k=0}^{\infty} a_k x^k \quad \text{Converges on } (-r, r)$$

then f is differentiable on $(-r, r)$

and

Converges to $f'(x)$

$$f'(x) = \underline{\sum_{k=0}^{\infty} \frac{d}{dx}(a_k x^k)}$$

$$= \sum_{k=1}^{\infty} k a_k x^{k-1} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$$

on $(-r, r)$.

Observation:

If

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = L \quad \left(\Rightarrow r = \frac{1}{L} \text{ is the radius of convergence of } \sum_{k=0}^{\infty} a_k x^k \right)$$

Then

$$\lim_{n\to\infty} \sqrt[n]{|n a_n|} = \lim_{n\to\infty} \underbrace{\sqrt[n]{n}}_{\to 1} \cdot \underbrace{\sqrt[n]{|a_n|}}_{\to L} = L$$

$$\left(\Rightarrow r = \frac{1}{L} \text{ is also the radius of convergence of } \sum_{k=1}^{\infty} k a_k x^{k-1} \right)$$

Example

Since

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots \quad \text{on } (-1, 1),$$

we have

① $$\sum_{k=0}^{\infty} (x^k)' = \sum_{k=1}^{\infty} k \cdot x^{k-1} = \left(\frac{1}{1-x}\right)'$$

$-(1-x)'$

1

$$= \frac{\quad}{(1-x)^2} = \boxed{\frac{\quad}{(1-x)^2}}$$

on $(-1, 1)$

② $$\sum_{k=1}^{\infty} (k \cdot x^{k-1})' = \sum_{k=2}^{\infty} k(k-1) \cdot x^{k-2}$$

$$= \left(\frac{1}{(1-x)^2}\right)' = -2 \cdot (1-x)^{-3} \cdot (-1)$$

$$= \boxed{\frac{2}{(1-x)^3} = \sum_{k=2}^{\infty} k(k-1) x^{k-2}}$$

on $(-1, 1)$

Example

Recall

$$e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k$$

is the solution of

$\circledast$ $$f'(x) = f(x), \qquad f(0) = 1$$

One can try to find a sol by:

$$\left(\sum^{\infty} a_k x^k\right)' = \sum_{k=}^{\infty} a_k \boxed{x^k}$$

Thm 11

$$\sum_{k=0}^{\infty} (a_k x^k)' = \sum_{k=1}^{\infty} k a_k x^{k-1}$$

(let $n = k-1$)

$$= \sum_{\cancel{n}^{k}=0}^{\infty} (\cancel{n}^{k}+1)\, a_{\cancel{n}^{k}+1} \cdot \boxed{x^{\cancel{n}^{k}}}$$

$$\sum_{n=0}^{\infty} (n+1)\, a_{n+1}\, x^n$$

$$= 1\cdot a_1 \cdot x^0 + 2\cdot a_2\, x^1 + \cdots$$

$$= \sum_{k=0}^{\infty} (k+1)\, a_{k+1}\, x^k$$

$$a_k = (k+1)\, a_{k+1}$$

$$\Rightarrow a_{k+1} = \frac{1}{(k+1)} \cdot a_k$$

$k \to k-1$

$$\Rightarrow a_k = \frac{1}{k}\, a_{k-1}$$

$k \to k-2$

$$\Rightarrow a_{k+1} = \frac{1}{k+1} \cdot \frac{1}{k}\, a_{k-1}$$

$(k \geq 0)$

$$= \frac{1}{k+1} \cdot \frac{1}{k} \cdot \frac{1}{k-1}\, a_{k-2} = \cdots$$

$$= \frac{1}{(k+1)!}\, a_0$$

correction: $\frac{a_0}{k!}$

$$\Rightarrow f(x) = a_0 + \sum_{k=\cancel{0}\,1}^{\infty} \frac{1}{(k\cancel{+1})!}\, x^k$$

$f(0) = 1$

$\Rightarrow a_0 = 1$

$$= \sum_{k=0}^{\infty} \frac{1}{k!}\, x^k$$

is a sol. #

<u>Thm</u> (Thm 12.9.3)

If
$$f(x) = \sum_{k=0}^{\infty} a_k x^k \quad \text{converges on } (-r, r)$$

then
$$\int f(x)\,dx = \sum_{k=0}^{\infty} \int a_k x^k\,dx + C$$

$$= \sum_{k=0}^{\infty} a_k \frac{x^{k+1}}{k+1} + C$$

on $(-r, r)$

<u>Example</u>

① $$\sum_{k=1}^{\infty} \frac{x^k}{k} = ?$$

<u>Recall</u>:
$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \quad \text{on } (-1, 1)$$

$$\Rightarrow \int \frac{1}{1-x}\,dx = \sum_{k=0}^{\infty} \int x^k\,dx + C$$

$$= \sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1} + C$$

$$\left(\ln(1-x)\right)' = \frac{1}{1-x} \cdot (1-x)' = -\frac{1}{1-x}$$

$$\Rightarrow \left(-\ln(1-x)\right)' = \underline{\left(\ln\frac{1}{1-x}\right)' = \frac{1}{1-x}}$$

So
$$\ln\left(\frac{1}{1-x}\right) = \sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1} + C$$
on $(-1,1)$

<u>At $x=0$:</u>

$$\ln\left(\frac{1}{1-0}\right) = \ln 1 = 0$$
$$= \sum_{k=0}^{\infty} \frac{0^{k+1}}{k+1} + C = C$$

Therefore,

$$\ln\left(\frac{1}{1-x}\right) = \sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1}$$

$$= \frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{3} + \cdots$$

$$= \boxed{\sum_{k=1}^{\infty} \frac{x^k}{k} = \ln\left(\frac{1}{1-x}\right)}$$

on $(-1, 1)$ #

② Prove

$$\ln(1+x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} x^{k+1}$$

$\forall\ x \in (-1, 1)$

<u>pf</u>

Consider

$$\frac{1}{1+x} = \sum_{k=0}^{\infty} (-x)^k \quad \text{on } (-1, 1)$$

$\Rightarrow$

$$\int \frac{1}{1+x}\,dx = \sum_{k=0}^{\infty} \int (-1)^k x^k\,dx + C$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} x^{k+1} + C$$

$$\left(\ln(1+x)\right)' = \frac{1}{1+x} \cdot (1+x)' = \frac{1}{1+x}$$

$$\Rightarrow \quad \ln(1+x) = \sum_{k=0} \frac{(-1)}{k+1} x^{k+1} + C'$$

on $(-1, 1)$

At $x = 0$:

$$\ln(1+0) = 0$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} 0^{k+1} + C = C$$

$$\Rightarrow \quad C = 0$$

So

$$\ln(1+x) \overset{(**)}{=} \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} x^{k+1} \quad \forall x \in (-1, 1)$$

\#

About $(**)$, note at $x = 1$

$$\text{LHS} = \ln(1+1) = \ln 2$$

$$\text{RHS} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} 1^{k+1} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \quad \text{Converges}$$

Q:

$$\ln 2 \overset{?}{=} \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$$

A: They are equal

Thm (Thm 12.9.5)

Suppose $\sum_{k=0}^{\infty} a_k x^k$ converges on $(-r, r)$ and

$$f(x) = \sum_{k=0}^{\infty} a_k x^k \quad \text{on } (-r, r).$$

(i) If f is left continuous at r

(i.e. $\lim_{x \to r^-} f(x) = f(r)$) and

$\sum_{k=0}^{\infty} a_k r^k$ converges, then

$$f(r) = \sum_{k=0}^{\infty} a_k r^k$$

(ii) If f is right continuous at $-r$

and $\sum_{k=0}^{\infty} a_k (-r)^k$ converges, then

$$f(-r) = \sum_{k=0}^{\infty} a_k (-r)^k$$

Remark

power series

Consider

$$f(x) = \sum_{k=0}^{\infty} a_k x^k$$

→ function

→ Taylor series

If it converges on $(-r, r)$, then the Taylor series of $f(x)$ is $\sum_{k=0}^{\infty} a_k x^k$ which converges to $f(x)$ on $(-r, r)$

because

$$\frac{f^{(n)}(0)}{n!} = \frac{\sum_{k=0}^{\infty} \frac{d^n(a_k x^k)}{dx^n}}{n!}\Bigg|_{x=0}$$

$$= \sum_{k=n}^{\infty} \frac{k\cdot(k-1)\cdots(k-n+1)\, a_k\, x^{k-n}}{n!}\Bigg|_{x=0}$$

$$= \frac{n!}{n!} a_n = a_n$$

Example

① $\cosh x = \dfrac{e^x + e^{-x}}{2}$

$$(-1)^k \cdot x^k = \begin{cases} x^k & k \text{ even} \\ -x^k & k \text{ odd} \end{cases}$$

$$= \frac{1}{2}\left(\sum_{k=0}^{\infty} \frac{x^k}{k!} + \sum_{k=0}^{\infty} \frac{(-x)^k}{k!} \right)$$

$$= \frac{1}{\cancel{2}}\left(\sum_{k=0}^{\infty} \frac{\cancel{2}\cdot x^{2k}}{(2k)!}\right)$$

$$= \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} \quad \text{on} \quad (-\infty, \infty)$$

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② $x^2\cos(x^3)$

$$= x^2 \cdot \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} (x^3)^{2k}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{6k+2}$$

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Recall

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}$$

$\forall x \in (-\infty, \infty)$