Calculus 4/2

Recall

Consider

$$\sum_{k=0}^{\infty} a_k x^k$$

Where does it converge?

Thm

If $\sum_{k=0}^{\infty} a_k x^k$ converges at $x = c$, $(c \neq 0)$

then $\sum_{k=0}^{\infty} a_k x^k$ converges absolutely at all x with $|x| < |c|$.

From this thm, we know:

case1:

$\sum_{k=0}^{\infty} a_k x^k$ converges only at $x=0$.

case2:

$\sum_{k=0}^{\infty} a_k x^k$ converges absolutely at all real numbers x. For example, $\sum_{k=0}^{\infty} \frac{x^k}{k!}$.

<u>case3:</u>

There exists $r>0$ s.t.

(i) $\sum_{k=0}^{\infty} a_k x^k$ converges absolutely for $|x|<r$.

(ii) $\sum_{k=0}^{\infty} a_k x^k$ diverges for $|x|>r$.

For example, $\sum_{k=0}^{\infty} x^k \begin{cases} = \dfrac{1}{1-x} & |x|<1 \\ \text{diverges} & |x|>1 \end{cases}$

<u>Def</u> 收斂半徑

The radius of convergence of $\sum_{k=0}^{\infty} a_k x^k$ is

(i) 0 if it is <u>case1</u>,

(ii) ∞ if it is <u>case2</u>,

(iii) r if it is <u>case3</u>.

The interval of convergence of a power series is the maximal interval on which it converges.

Example

① Consider

$$\sum_{k=0}^{\infty} k^k x^k = \sum_{k=0}^{\infty} (kx)^k$$

∞ at each fixed nonzero x as $k\to\infty$

Since

$$\lim_{k\to\infty} (kx)^k \neq 0$$

at each nonzero x,

$$\sum_{k=0}^{\infty} k^k x^k \text{ diverges. } \forall x \neq 0.$$

$\Rightarrow$ The radius of convergence $= 0$. #

② The radius of convergence of

$$\sum_{k=0}^{\infty} \frac{1}{k!} x^k \quad (\text{Converges to } e^x \ \forall x)$$

is ∞.

③ The radius of convergence of

$$\sum_{k=0}^{\infty} x^k \quad \begin{cases} \text{Converges} & \forall |x| < 1 \\ \text{diverges} & \forall |x| \geq 1 \end{cases}$$

is 1

① interval of convergence of

$$\sum_{k=0}^{\infty} k^k x^k$$

is $\{0\}$

② interval of convergence of

$$\sum_{k=0}^{\infty} \frac{x^k}{k!}$$

is $(-\infty, \infty)$

③ interval of convergence of

$$\sum_{k=0}^{\infty} x^k$$

is $(-1, 1)$

Note:

$x=1$: $\sum_{k=0}^{\infty} 1^k$ diverges

$x=-1$: $\sum_{k=0}^{\infty} (-1)^k$ diverges

Remark

It is typical to find the radius of convergence by root test or ratio test.

(i) Root test:

① $\sum_{k=0}^{\infty} c_k x^k$ converges absolutely if

① $\sum_{k=0} a_k x^k$ converges absolutely if

$$\boxed{\lim_{k\to\infty} \sqrt[k]{|a_k x^k|}} < 1$$

$$= \lim_{k\to\infty} \sqrt[k]{|a_k|} \cdot \underbrace{\boxed{\sqrt[k]{|x|^k}}}_{=|x|} = \left(\lim_{k\to\infty} \sqrt[k]{|a_k|}\right) \cdot |x|$$

$$\Leftrightarrow \quad |x| < \frac{1}{\lim_{k\to\infty} \sqrt[k]{|a_k|}} \quad \left(\text{if } \lim_{k\to\infty} \sqrt[k]{|a_k|} \text{ exists}, \neq 0\right)$$

② $\sum_{k=0}^{\infty} a_k x^k$ diverges if

$$\lim_{k\to\infty} \sqrt[k]{|a_k x^k|} > 1$$

$$= \left(\lim_{k\to\infty} \sqrt[k]{|a_k|}\right) \cdot |x|$$

$$\Leftrightarrow \quad |x| > \frac{1}{\lim_{k\to\infty} \sqrt[k]{|a_k|}}$$

Conclusion:

If $\lim_{k\to\infty} \sqrt[k]{|a_k|}$ exists, $= L$, then

$r = \frac{1}{L}$ is the radius of convergence.

(ii) Ratio Test:

$$\lim_{k\to\infty}\left|\frac{a_{k+1}\cancel{x^{k+1}}^{\,x}}{a_k\cancel{x^k}}\right| = \left(\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right|\right)\cdot|x| \quad \begin{matrix}<1\\ \text{or}\\ >1\end{matrix}$$

By the same argument, we have that

$$r = \frac{1}{\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right|}$$

$\Rightarrow$ converges when $|x|<r$, diverges when $|x|>r$

is the radius of convergence of $\sum_{k=0}^{\infty} a_k x^k$. (Assume $\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right|$ exists.)

Example

Find the interval of convergence of the series.

Step 1: Find the radius of convergence (assum it's r)

Step 2: Check the convergence of the power series at $x=\pm r$.

① $\sum_{k=1}^{\infty}\frac{x^k}{k}$

Step1 :

$$\lim_{k\to\infty} \sqrt[k]{|a_k|} = \lim_{k\to\infty} \sqrt[k]{\frac{1}{k}} = \lim_{k\to\infty} \frac{1}{\sqrt[k]{k}} \to 1$$

$$= 1. \Rightarrow r = \frac{1}{1} = 1$$

Step2 :

$x=1$: $\sum_{k=1}^{\infty} \frac{1^k}{k} = \sum_{k=1}^{\infty} \frac{1}{k}$ diverges

$x=-1$: $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ Converges $\left(\sum_{k=1}^{\infty} (-1)^k \cdot a_k\right)$

Conclusion :

The interval of convergence of

$\sum_{k=1}^{\infty} \frac{x^k}{k}$ is $[-1, 1)$. #

② $\sum_{k=1}^{\infty} \frac{x^k}{k^2}$: $(a_k = \frac{1}{k^2}$

Step1 :

$$r = \frac{1}{\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right|} = \frac{1}{\lim_{k\to\infty} \frac{\left(\frac{1}{k+1}\right)^2}{\left(\frac{1}{k}\right)^2}}$$

$$= \frac{1}{\ldots} = 1$$

$\lim_{k\to\infty} \left(\frac{\quad}{k+1}\right)$ 1

<u>Step 2</u>:

<u>$x=1$</u>: $\sum_{k=1}^{\infty} \frac{1}{k^2}$ Converges

<u>$x=-1$</u>: $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$ Converges

<u>Conclusion</u>

interval of Convergence of $\sum_{k=1}^{\infty} \frac{x^k}{k^2}$

is $[-1, 1]$ #

③ $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2 3^k} (x+2)^k$:

<u>Step 1</u>:

radius of convergence

$$= \frac{1}{\lim_{k\to\infty} \sqrt[k]{\left|\frac{(-1)^k}{k^2\, 3^k}\right|}} = \frac{1}{\lim_{k\to\infty} \frac{1}{\boxed{\sqrt[k]{k}}^{\,2} \cdot \sqrt[k]{3^k}}}$$

$\sqrt[k]{k} \to 1$, $\sqrt[k]{3^k} = 3$

$$= 3$$

$-2-3 < x < -2+3$

$$\Rightarrow \begin{cases} |x+2| < 3 \Rightarrow \text{converge} \\ |x+2| > 3 \Rightarrow \text{diverge} \end{cases}$$

-5

<u>Step2</u>

<u>$x=-5$</u> : $\displaystyle\sum_{k=1}^{\infty} \frac{\cancel{\boxed{(-1)^k}}}{k^2 \cdot \cancel{\boxed{3^k}}} (-5+2)^k = \sum_{k=1}^{\infty} \frac{1}{k^2}$

$\cancel{(-3)^k}$

Converges

<u>$x=1$</u> : $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2 \cdot \cancel{3^k}} \cancel{(1+2)^k} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$

Converges

<u>Conclusion</u>

interval of convergence of $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2 3^k} (x+2)^k$

is $[-5, 1]$ #