Calculus 3/26

<u>Recall</u>

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} \qquad \forall x \in (-\infty, \infty)$$

<u>Exercise</u>:

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

Consider Taylor series $x-a$:

<u>Thm</u> (Thm 12.7.1)

If f has $n+1$ continuous derivatives on an open interval I that contains a, then for each $x \in I$,

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x),$$

where

$$R_n(x) = \frac{1}{n!}\int_a^x f^{(n+1)}(t)\cdot(x-t)^n\,dt.$$

depending on x

Furthermore, there exists a number c between a and x s.t.

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

Example

The polynomial

$= a(x-2)^3 + b(x-2)^2 + c(x-2)^1 + d$

$$f(x) = 4x^3 - 3x^2 + 5x - 1$$

can be expanded in powers of $x-2$ by computing $f^{(n)}(2)$:

$$f(2) = 4\times 8 - 3\times 4 + 5\times 2 - 1 = 29$$

$$f'(2) = 12x^2 - 6x + 5\Big|_{x=2}$$

$$= 12\times 4 - 6\times 2 + 5 = 41$$

$$f''(2) = 42$$

$$f'''(2) = 24$$

$$f^{(n)}(2) = 0 \quad \forall n \geq 4$$

So

$$f(x) = f(2) + f'(2)\cdot(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3 + \cancel{\boxed{\frac{f^{(4)}(c)}{4!}(x-2)^4}} \Rightarrow 0$$

$$= 29 + 41(x-2) + 21(x-2)^2 + 4(x-2)^3 \quad \#$$

§ Power series

We approach this problem from another direction: Consider

power series

$$\underline{\sum_{k=0}^{\infty} a_k x^{\boxed{k}}} \quad \text{and} \quad \sum_{k=0}^{\infty} a_k \cdot (x-a)^k$$

without a given function $f(x)$.

Def

A power series $\sum_k a_k x^k$

(i) Converges at c if $\sum_k a_k c^k$ converges

(ii) Converges on a set S if $\sum_k a_k x^k$ converges at each $x \in S$.

Example

① $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ converges on $(-\infty, \infty)$.

② $\sum_{k=0}^{\infty} x^k$ converges on $(-1, 1)$

Remark

$\sum_{k=0}^{\infty} a_k x^k$ always converges at $x=0$

because $\sum_{k=0}^{\infty} a_k 0^k = a_0$

Main question:

At what x, does $\sum_{k=0}^{\infty} a_k x^k$ converge?

Thm (Thm 12.8.2)

(i) If $\sum_k a_k x^k$ converges at $c \neq 0$, then it converges absolutely at all x with $|x| < |c|$.

(ii) If $\sum a_k x^k$ diverges at d, then it diverges at all x with $|x| > |d|$.

pf

Suppose $\sum a_k c^k$ converges.

$\Rightarrow \lim_{k\to\infty} a_k c^k = 0$

$\Rightarrow$ For k sufficiently large,

$$|a_k c^k| < 1.$$

$\Rightarrow \forall x$ with $|x| < |c|$, k sufficiently large

$$0 \leq |a_k x^k| = \underbrace{|a_k c^k|}_{<1} \cdot \left|\frac{x}{c}\right|^k < \left|\frac{x}{c}\right|^k$$

Since

Since $\left|\frac{\hat{x}}{c}\right| < 1 \Rightarrow \sum_k \left|\frac{\hat{x}}{c}\right|^k$ converges

by the Comparison thm,

$$\sum |a_k x^k| \quad \text{converges}$$

$$\Rightarrow \quad \sum a_k x^k \quad \text{is absolutely convergent} \quad \#$$

HW2. 1. (i)

$$a_n = \left(\frac{1}{2} + \frac{3}{n}\right)^{3n}$$

sol :

For $n \geq 9$,

$$\frac{1}{2} + \frac{3}{n} \leq \frac{1}{2} + \frac{3}{9} = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$$

$$\Rightarrow \quad 0 \leq \left(\frac{1}{2} + \frac{3}{n}\right)^{3n} \leq \left(\underbrace{\frac{5}{6}}_{<1}\right)^{3n} \longrightarrow 0 \text{ as } n \to \infty$$

By the pinching thm,

$$\lim \left(\frac{1}{2} + \frac{3}{n}\right)^{3n} = 0 \quad \#$$

$\lim_{n\to\infty}$

HW3.

等比級數 Converges because $\left|-\frac{1}{5}\right| = \frac{1}{5} < 1$

(b) $$\sum_{k=0}^{\infty} \frac{(-1)^k}{5^k} = \sum_{k=0}^{\infty} \left(\frac{-1}{5}\right)^k$$

$$= \frac{1}{1-\left(-\frac{1}{5}\right)} = \frac{5}{6} \quad \#$$

(C) $$\sum_{k=0}^{\infty} \frac{3^{k-1}}{4^{3k+1}} = \sum_{k=0}^{\infty} \frac{3^{-1}}{4} \cdot \frac{3^k}{4^{3k}}$$

$$= \frac{1}{12} \cdot \sum_{k=0}^{\infty} \left(\frac{3}{4^3}\right)^k = \frac{1}{12} \sum_{k=0}^{\infty} \left(\frac{3}{64}\right)^k$$

$$= \cdots$$

HW2. 4. (b)

$$\int_{-\infty}^{0} xe^x \, dx = \lim_{b\to-\infty} \int_b^0 xe^x \, dx$$

$\int u' \cdot v \, dx = uv - \int u \cdot v' \cdot dx$

$$\int_b^0 xe^x dx = xe^x \Big|_b^0 - \int_b^0 e^x dx$$

$u' = e^x,\ u = e^x$

$v = x,\ v' = 1$

$$= -be^b - e^x \Big|_b^0$$

$$= -be^{b} - 1 + e^{b}$$

$$\text{ans} = \lim_{b \to -\infty} \left(-be^{b} - 1 + e^{b}\right)$$

$c = -b$

$$= \lim_{c \to +\infty} \left(ce^{-c} - 1 + e^{-c}\right)$$

$$= \lim_{c \to \infty} \left(\boxed{\frac{c}{e^{c}} + \frac{1}{e^{c}}}^{\to 0} - 1 \right) = -1$$

#

HW3. 5. (c)

$$\sum \frac{\ln k}{k^2} \longleftrightarrow \sum \frac{1}{k^p} \begin{cases} \text{converges } \forall p > 1 \\ \text{diverges } p \leq 1 \end{cases}$$

Choose $p = \frac{3}{2}$

$$\lim_{k \to \infty} \frac{\frac{\ln k}{k^2}}{\frac{1}{k^{\frac{3}{2}}}} = \lim_{k \to \infty} \frac{\ln k \to \infty}{\sqrt{k} \to \infty}$$

$$= \lim_{k \to \infty} \frac{\frac{1}{k}}{\frac{1}{2} \cdot \frac{1}{\sqrt{k}}} = \lim_{k \to \infty} \frac{2}{\sqrt{k}} = 0$$

$\Rightarrow$ For k sufficiently large, (consider $k > 1$)

$$0 \leq \frac{\ln k}{k^2} \leq \frac{1}{k^{\frac{3}{2}}}$$

Since $\sum \frac{1}{k^{\frac{3}{2}}}$ converges, by the comparison test, $\sum \frac{\ln k}{k^2}$ converges #

HW2. 4. (d)

$$\int_{-\infty}^{\infty} \frac{1}{e^x + e^{-x}}\,dx = \int_{-\infty}^{0} \frac{1}{e^x+e^{-x}}\,dx + \int_{0}^{\infty} \frac{1}{e^x+e^{-x}}\,dx$$

$$\int_0^\infty \frac{1}{e^x+e^{-x}}\,dx = \lim_{a\to\infty} \int_0^a \frac{1}{e^x+e^{-x}}\,dx \qquad = \frac{e^x}{e^{2x}+1}$$

$$= \lim_{a\to\infty} \int_0^a \frac{1}{\underbrace{(e^x)^2}_{u^2}+1} \underbrace{e^x\,dx}_{du} \qquad u = e^x \quad du = e^x\,dx$$

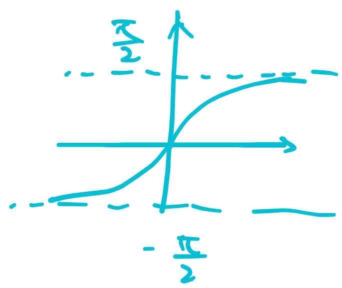
$$= \lim_{a\to\infty} \int_1^{e^a} \frac{1}{u^2+1}\,du$$

$$= \lim \tan^{-1} u \Big|^{e^a} = \lim \underbrace{\tan^{-1} e^a}_{\frac{\pi}{2}} - \underbrace{\tan^{-1} 1}_{\frac{\pi}{4}}$$

$\lim_{a\to\infty}$ $\quad = \quad$ $\lim_{a\to\infty}$

$$= \frac{\pi}{4}$$

y = tan⁻¹x



$$\int_{-\infty}^{0} \frac{1}{e^{x}+e^{-x}}\,dx$$

$$= \lim_{b\to-\infty} \boxed{\int_{b}^{0} \frac{1}{e^{x}+e^{-x}}\,dx} = \int_{b}^{0} \frac{e^{x}}{(e^{x})^{2}+1}\,dx$$

$$= \int_{e^{b}}^{1} \frac{du}{u^{2}+1}$$

$$= \tan^{-1}u \Big|_{e^{b}}^{1}$$

$$= \frac{\pi}{4} - \tan^{-1}e^{b}$$

$$= \lim_{b\to-\infty}\left(\frac{\pi}{4} - \tan^{-1}\overset{0}{\overset{\uparrow}{e^{b}}}\right)$$

$$= \frac{\pi}{4}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{1}{e^{x}+e^{-x}}\,dx = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}. \quad \#$$