Calculus 3/21

<u>Thm</u> (Taylor's thm)

If f has $n+1$ continuous derivatives on an open interval I that contains 0, then for each $x \in I$

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots$$

(*)
$$+ \frac{f^{(n)}(0)}{n!}x^n + R_n(x)$$

$$\frac{f^{(n+1)}(\boxed{c})}{(n+1)!}x^{n+1}$$

depending on x

$c = c(x)$

where

$$R_n(x) = \frac{1}{n!}\int_0^x f^{(n+1)}(t)\cdot(x-t)^n\,dt$$

<u>pf</u> (we are using induction)

Fix $x \in I$. Since

$$\int_0^x f'(t)\,dt = f(x) - f(0),$$

we have "poly of deg 0" $R_0(x)$

$$f(x) = \underline{f(0)} + \underline{\int_0 f'(t)\cdot 1\,dt}$$

—— ⊛ with $n=0$.

integrate it by parts

$$\int u(t)\cdot v'(t)\,dt = u(t)\cdot v(t) - \int v(t)\,u'(t)\,dt$$

Let $u(t) = f'(t)$, $\quad v(t) = t-x$

$\quad u'(t) = f''(t)$, $\quad v'(t) = 1$

$$\cancel{f'(x)\cdot(x-x)} \; {=0}$$
$$-\,f'(0)\cdot(0-x)$$
$$= f'(0)\cdot x$$

Then

$$R_0(x) = \int_0^x f'(t)\cdot 1\,dt = \boxed{f'(t)\cdot(t-x)\Big|_{t=0}^{x}} - \int_0^x f''(t)\cdot(t-x)\,dt$$

$$= f'(0)\cdot x + \int_0^x f''(t)\cdot(x-t)\,dt$$

$$\Rightarrow \quad f(x) = f(0) + f'(0)\cdot x + \underbrace{\int_0^x f''(t)\cdot(x-t)\,dt}_{R_1(x)}$$

—— ⊛ with $n=1$.

$\vdots$

Assume we have

Assume we have

$$f(x) = f(0) + f'(0)x + \cdots + \frac{f^{(k)}(0)}{k!}x^k + R_k(x)$$

Then, by taking

$$u(t) = f^{(k+1)}(t) \qquad v(t) = \frac{-(x-t)^{k+1}}{(k+1)!}$$

$$u'(t) = f^{(k+2)}(t) \qquad v'(t) = \frac{(x-t)^k}{k!}$$

$$R_k(x) = \frac{1}{k!}\int_0^x \underline{f^{(k+1)}(t)} \cdot \underline{(x-t)^k}\, dt$$

$$= \boxed{f^{(k+1)}(t) \cdot \frac{-(x-t)^{k+1}}{(k+1)!}\Bigg|_{t=0}^{x}}$$

$$= f^{(k+1)}(x) \cdot \cancel{\frac{-(x-x)^{k+1}}{(k+1)!}} \; {=0}$$

$$- f^{(k+1)}(0) \cdot \frac{-(x-0)^{k+1}}{(k+1)!}$$

$$= f^{(k+1)}(0) \cdot \frac{x^{k+1}}{(k+1)!}$$

$$+ \int_0^x f^{(k+2)}(t) \cdot \left(\frac{+(x-t)^{k+1}}{(k+1)!} \right) dt$$

$$= \frac{f^{(k+1)}(0)}{(k+1)!} \cdot x^{k+1} + \boxed{\int_0^x f^{(k+2)}(t) \cdot \frac{(x-t)^{k+1}}{(k+1)!}\, dt}$$

$$\boxed{\int_0^x f^{(k+2)}(t) \cdot \frac{(x-t)^{k+1}}{(k+1)!}\, dt} = \frac{1}{(k+1)!}\int_0^x f^{(k+2)}(t) \cdot (x-t)^{k+1}\, dt = R_{k+1}(x)$$

$$= R_k(x)$$

Therefore,

induction hypothesis

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(k)}(0)}{k!}x^k + \underline{\underline{R_k(x)}}$$

our computation

$$= f(0) + f'(0)x + \cdots + \frac{f^{(k)}(0)}{k!}x^k + \frac{f^{(k+1)}(0)}{(k+1)!}x^{k+1} + R_{k+1}(x)$$

———— ⊛ with $n = k+1$

(induction)

This proves Taylor's Thm. #

The remainder $R_n(x)$ can be expressed in another form:

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}$$

with c some number between 0 and x.

To get this expression, we need

2nd Mean Value Thm:

<u>Thm</u> (2nd MVT for integrals, Thm 5.9.3)

If u and v are continuous on $[a,b]$ and $v \geq 0$ on $[a,b]$, then there exists $c \in [a,b]$ s.t.

$$\int_a^b u(x) \cdot v(x)\,dx = u(c) \cdot \int_a^b v(x)\,dx$$

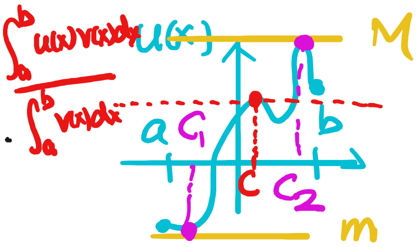
Such $u(c)$ is called the v-weight average of u on $[a,b]$.

pf

If $v(x)$ is the zero function on $[a,b]$ then the thm is true.

Assume $v(x)$ is NOT the zero function.

$$\Rightarrow \quad \int_a^b v(x)\,dx > 0.$$



Since u is continuous on $[a,b]$, by the extreme value thm, $\exists\, m, M$ s.t.

$$m \leq u(x) \leq M, \quad \forall x \in [a,b]$$

Since $v(x) \geq 0$,

$$m \cdot v(x) \leq u(x) \cdot v(x) \leq M \cdot v(x)$$

$$\Rightarrow \quad m \cdot \underbrace{\int_a^b v(x)\,dx}_{>0} \leq \int_a^b u(x) \cdot v(x)\,dx \leq M \cdot \underbrace{\int_a^b v(x)\,dx}_{>0}$$

$$\Rightarrow \quad \underset{u(c_1)}{\overset{\shortparallel}{m}} \leq \frac{\int_a^b u(x) \cdot v(x)\,dx}{\int_a^b v(x)\,dx} \leq \underset{u(c_2)}{\overset{\shortparallel}{M}}$$

By the intermediate value thm,
there exists c between c_1 and c_2
s.t.

$$u(c) = \frac{\int_a^b u(x)\, v(x)\, dx}{\int_a^b v(x)\, dx}$$

$$\Rightarrow \quad \int_a^b u(x)\, v(x)\, dx = u(c) \cdot \int_a^b v(x)\, dx$$

#

Recall

$$R_n(x) = \frac{1}{n!} \int_0^x f^{(n+1)}(t) \cdot (x-t)^n \, dt$$

Apply 2nd MVT to $R_n(x)$ with
(if $x \geq 0$)

$$u(t) = f^{(n+1)}(t)$$

(**) $\left\{ V(t) = \dfrac{(x-t)^n}{n!} \geq 0 \quad \forall t \in [0,x] \right.$

Then $\exists\, c \in [0, x]$ s.t.

$$\int_0^x u(t)\cdot V(t)\, dt = R_n(x)$$

$$\overset{\text{2nd MVT}}{=} u(c)\cdot \int_0^x V(t)\,dt$$

$$\overset{(**)}{=} f^{(n+1)}(c)\cdot \boxed{\int_0^x \frac{(x-t)^n}{n!}\, dt}$$

$$\left((x-t)^{n+1}\right)' = (n+1)\cdot(x-t)^n \cdot \underbrace{\boxed{(x-t)'}}_{-1}$$

$$\| $$

$$\left.\frac{(x-t)^{n+1}}{(n+1)!}(-1)\right|_{t=0}^{x}$$

$$\|$$

$$(-1)\cancel{\frac{(x-x)^{n+1}}{(n+1)!}}^{=0} - (-1)\frac{(x-0)^{n+1}}{(n+1)!}$$

$$\|$$

$$\frac{x^{n+1}}{(n+1)!}$$

$$= \boxed{f^{(n+1)}(c)\; x^{n+1} \quad R \ldots}$$

$$- \frac{\quad}{(n+1)!} x \qquad = R_n(x)$$

Corollary = 推論

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The remainder

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}$$

for some number c between 0 and x.

In other words,

$$\boxed{f(x) = f(0) + f'(0)\cdot x + \frac{f''(0)}{2!} x^2 + \cdots + \frac{f^{(n)}(0)}{n!} x^n + \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}}$$

depending on x

$c(x)$

with $\boxed{c}$ some number between 0 and x.

Remark

$$|R_n(x)| = \left| \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1} \right|$$

Recall: $\frac{a^n}{n!} \to 0$

$$= \left| f^{(n+1)}(c) \right| \cdot \frac{|x|^{n+1}}{(n+1)!}$$

(if $x \geq 0$)

$$\leq \left(\max_{t \in [0,x]} \left| f^{(n+1)}(t) \right| \right) \cdot \frac{|x|^{n+1}}{(n+1)!}$$

0 as $n \to \infty$

This inequality sometimes helps us deduce

$$\lim_{n \to \infty} R_n(x) = 0 .$$

<u>Example</u>

Consider

$$f(x) = e^x \Rightarrow f^{(n)}(x) = e^x$$

$$f(0) = e^0 = 1$$

$$f'(0) = e^0 = 1$$

$$\vdots$$

$$f^{(n)}(0) = e^0 = 1$$

$\Rightarrow$

$$e^x = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n$$

$$= \overbrace{1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}}^{\sum_{k=0}^{n} \frac{x^k}{k!}} + R_n(x)$$

And

$$0 \leq |R_n(x)| \leq \left(\max_{\substack{t\in[0,x] \\ \text{or} \\ t\in[x,0]}} \underbrace{|f^{(n+1)}(t)|}_{= |e^t|} \right) \cdot \frac{|x|^{n+1}}{(n+1)!}$$

$$\leq \underbrace{e^{|x|}}_{\text{does NOT depend on } n} \cdot \frac{|x|^{n+1}}{(n+1)!} \longrightarrow 0 \quad \text{as } n\to\infty$$

By the pinching thm,

$$\lim_{n\to\infty} R_n(x) = 0 \qquad \forall x.$$

$$\| $$

$$\lim_{n\to\infty} \left(e^x - \sum_{k=0}^{n} \frac{x^k}{k!} \right)$$

$$\| $$

$$e^x - \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

So

$$\boxed{\begin{aligned} e^{x} &= \sum_{k=0}^{\infty} \frac{x^k}{k!} \qquad \forall x \in (-\infty, \infty) \\ &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \end{aligned}}$$

#

Example

Consider

$$f(x) = \sin x \xrightarrow{x=0} 0$$

$\Rightarrow$

$$f'(x) = \cos x, \quad f''(x) = -\sin x, \quad f'''(x) = -\cos x, \quad f^{(4)}(x) = \sin x \quad \cdots$$

$x=0$

$$1 \qquad 0 \qquad -1 \qquad 0$$

$\Rightarrow$

$$\sin x = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + R_n(x)$$

$$= \underbrace{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{f^{(n)}(0)}{n!}x^n}_{} + R_n(x)$$

$\lceil 2.12 \rceil = 2 \quad \rightsquigarrow \lceil \frac{n}{2} \rceil$

$$\sum_{k=0} \frac{(-1)}{(2k+1)!} x^{2k+1}$$

≤ 1

$$|R_n(x)| \leq \Big(\underbrace{\max_{\substack{t\in[0,x] \\ \text{or} \\ t\in[x,0]}} |f^{(n)}(t)|}_{} \Big) \cdot \frac{|x|^{n+1}}{(n+1)!}$$

$$\leq \frac{|x|^{n+1}}{(n+1)!} \longrightarrow 0 \quad \text{as } n\to\infty$$

$$\Rightarrow \quad \lim_{n\to\infty} R_n(x) = 0 \quad \forall\, x \in (-\infty, \infty)$$

$$\Rightarrow \quad \boxed{\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} \quad \forall\, x\in(-\infty,\infty) \\ = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots}$$

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