Calculus 3/19

Example

Consider

$$\sum_{k=1}^{\infty} \frac{k^k}{k!} \quad (= a_k)$$

Ratio test:

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{\frac{(k+1)^{k+1}}{(k+1)!}}{\frac{k^k}{k!}}$$

$$= \lim_{k\to\infty} \frac{\overset{\cancel{1\cdot2\cdot3\cdots k}}{k!}}{\underset{\cancel{1\cdot2\cdot3\cdots k}\cdot\cancel{(k+1)}}{(k+1)!}} \cdot \frac{\boxed{\cancel{(k+1)}}^{k+1}}{k^k} \qquad (k+1)^k$$

$$= \lim_{k\to\infty} \left(\frac{k+1}{k}\right)^k$$

Recall

$$\lim_{k\to\infty} \left(1 + \frac{x}{k}\right)^k = e^x$$

$$= \lim_{k\to\infty} \left(1 + \frac{1}{k}\right)^k = e > 1$$

So $\displaystyle\sum_{k=1}^{\infty} \frac{k^k}{k!}$ diverges. #

Next issue:

What if "$a_k > 0 \ \forall k$" is NOT satisfied?

Def

絕對收斂

A series $\displaystyle\sum_{k=1}^{\infty} a_k$ is absolutely convergent if the series $\displaystyle\sum_{k=1}^{\infty} |a_k|$ converges.

A series is conditionally convergent if it is convergent and NOT absolutely convergent.

Thm (Thm 12.5.1)

Absolutely convergent series are convergent.

Pf

Let $\sum_{k=1}^{\infty} a_k$ be an absolutely convergent series, i.e. $\sum_{k=1}^{\infty} |a_k|$ converges.

$\Rightarrow$ $\sum_{k=1}^{\infty} 2 \cdot |a_k|$ converges

$\sum_{k=1}^{\infty} a_k + |a_k|$

Since

$$0 \leq a_k + |a_k| \leq 2 \cdot |a_k| \quad \forall k,$$

by the basic comparison test.

$$\sum_{k=1}^{\infty} (a_k + |a_k|)$$

converges.

converges　　converges

$$\Rightarrow \sum^{\infty} a_k = \sum_{k=1}^{\infty} (a_k + |a_k|) - \sum_{k=1}^{\infty} |a_k|$$

$k=1$

also Converges #

<u>Example</u>

① $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} = 1 - \frac{1}{4} + \frac{1}{9} - \cdots$

is absolutely convergent because

$$\sum_{k=1}^{\infty} \left| \frac{(-1)^{k+1}}{k^2} \right| = \sum_{k=1}^{\infty} \frac{1}{k^2}$$

Converges

<u>Recall</u>:
p-series:
$\sum_{k=1}^{\infty} \frac{1}{k^p}$ Converges
if $p>1$

<u>Thm</u> $\Rightarrow$ $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2}$ Converges. #

②
$$\left(\frac{1}{2} - \frac{1}{2^2} - \frac{1}{2^3}\right) + \left(\frac{1}{2^4} - \frac{1}{2^5} - \frac{1}{2^6}\right) + \cdots$$

is absolutely convergent series because

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots$$

Converges.

Thm $\Rightarrow$ The given series converges. #

③ $$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

is NOT absolutely convergent because

$$\sum_{k=1}^{\infty} \left| \frac{(-1)^{k+1}}{k} \right| = \sum_{k=1}^{\infty} \frac{1}{k}$$

diverges. #

Q:

Does $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ converge?

Thm (Alternating series, Thm 12.5.3)

Let $(a_k)_{k=1}^{\infty}$ be a descending sequence of positive numbers. If

$$\lim_{k \to \infty} a_k = 0,$$

then

$$\sum_{k=1}^{\infty}(-1)^{k+1}a_k = a_1 - a_2 + a_3 - a_4 + \cdots$$

Converges.

<u>idea of pf</u>

Let $S_n = \sum_{k=1}^{n}(-1)^{k+1}a_k$

m: $\underbrace{S_{2m+1}}_{>0} = (\underbrace{a_1 - a_2}_{>0}) + (\underbrace{a_3 - a_4}_{>0}) + (\underbrace{\cdots}_{>0}) (\underbrace{\quad - a_{2m}}_{>0}) + \underbrace{a_{2m+1}}_{>0}$

m+1: $S_{2m+3} = a_1 - a_2 + \cdots + a_{2m+1}(-a_{2m+2} + a_{2m+3})$

$= S_{2m+1} - (\underbrace{a_{2m+2} - a_{2m+3}}_{>0})$

because (a_k) is decreasing

So $(S_{2m+1})_{m=1}^{\infty}$ is decreasing and bounded below by 0.

$\Rightarrow$ $(S_{2m+1})_m$ converges to some number L.

Since

$$S_{2m} = S_{2m-1} - a_{2m}$$

and

$$\lim_{m\to\infty} a_{2m} = 0,$$

we have

$$\lim_{m\to\infty} S_{2m} = \lim_{m\to\infty} S_{2m-1} = L$$

Therefore,

$$\sum_{k=1}^{\infty} (-1)^{k+1} a_k = \lim_{n\to\infty} S_n = L$$

Converges. #

<u>Remark</u>

A "rearrangement" of a series $\sum_{k=1}^{\infty} a_k$ is a series that has exactly same terms but in a different order. For example,

$$a_1 + a_3 + a_2 + a_5 + a_7 + a_4 + \cdots$$

is a rearragement

$$a_1 + a_2 + a_3 + a_4 + a_5 + \cdots$$

It is a theorem of Riemann:

(i) All rearrangements of an absolutely convergent series still converge to the same limit.

(ii) In a sharp contrast, a series that is conditionally convergent can be rearranged (1) to converge to ANY real number (2) to diverge to [illegible] number, ② to diverge to $+\infty$,

③ to diverge to $-\infty$, ④ or even to oscillate between any two numbers we choose.

§ Taylor expansion

We have understood the differentiation and integration of polynomials.

Now we try to approach more complicated functions by polynomials.

$f(x)$

<u>Hope</u>:

"similar" ??

$P(x)$

$$f(x) \approx a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

Let us assume they have the same derivatives up to the n-th order at $x=0$:

← Hope they look similar near $x=0$.

0-th order: $f(0) = P(0) = a_0$

1-st order: $f'(0) = P'(0) =$

$$a_1 + 2a_2 x + 3a_3 x^2 + \dots + n a_n x^{n-1} \Big|_{x=0}$$

$$= a_1$$

2nd order: $f''(0) = P''(0)$

$$= 2a_2 + 6a_3 x + \dots + n(n-1) a_n x^{n-2} \Big|_{x=0}$$

$$= 2a_2 = 2! \cdot a_2$$

i.e. $a_2 = \dfrac{f''(0)}{2!}$

$\vdots$

n-th order: $f^{(n)}(0) = n! \cdot a_n$

i.e. $a_n = \dfrac{f^{(n)}(0)}{n!}$

We consider

$\approx f(x)$ around $x=0$

$$P(x) = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \dots + \frac{f^{(n)}(0)}{n!} x^n$$

as the best degree-n polynomial approximation of $f(x)$ around $x=0$.

Consider large n. Consider

$$f(x) \approx \boxed{\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k}$$

$$= f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \cdots + \frac{f^{(k)}(0)}{k!} x^k + \cdots$$

as a good approximation of $f(x)$ around $x=0$.

<u>Questions:</u>

(i) For what x, does the series $\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$ converge?

(ii) If the series converge at x, does it really converge to $f(x)$?

<u>Thm</u> (Taylor's Thm, Thm 12.6.1)

If f has $n+1$ continuous derivatives on an open interval I that contains 0, then for each $x \in I$,

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + R_n(x),$$

where

$$R_n(x) = \frac{1}{n!}\int_0^x f^{(n+1)}(t) \cdot (x-t)^n \, dt$$

is called the remainder