# Calculus 3/14

**Recall**

① $\sum_{k=0}^{\infty} a_k = a_0 + a_1 + a_2 + \cdots$

$$= \lim_{n\to\infty} \sum_{k=0}^{n} a_k$$

② $\sum_{k=0}^{n} x^k = 1 + x + x^2 + x^3 + \cdots$

$$= \begin{cases} \dfrac{1}{1-x} & |x| < 1 \\ \text{diverges} & |x| \geq 1 \end{cases}$$

③ If $\sum_{k=0}^{\infty} a_k$ converges, then

$$\lim_{k\to\infty} a_k = 0.$$

**Note**

$\sum^{\infty} \frac{1}{\ }$ diverges and

$\sum_{k=1} \frac{1}{\sqrt{k}}$ diverges, and

$$\lim_{k\to\infty} \frac{1}{\sqrt{k}} = 0$$

## § Convergence of series

We will introduce many criteria for convergence of series:

Recall

If $(a_n)_{n\geq 1}^{\infty}$ is increasing and bounded above, then it converges.

Thm (Thm 12.3.1)

Assume $a_k \geq 0$, $\forall k$.

Then $\sum_{k=0}^{\infty} a_k$ converges if and only if $\left(\sum_{k=0}^{n} a_k\right)_{n=0}^{\infty}$ is bounded above.

Thm (Integral test Thm 12.3.2)

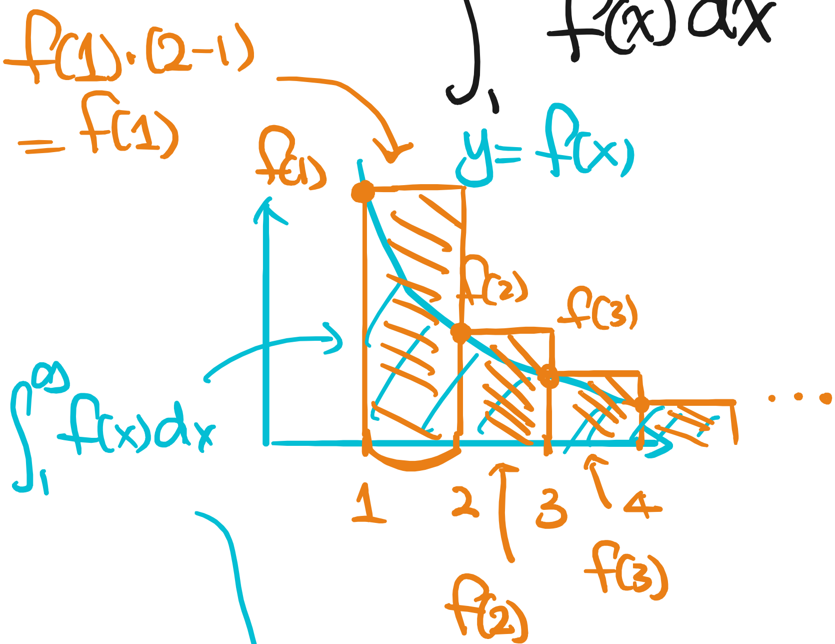
Assume $f$ is continuous, positive and decreasing on $[1,\infty)$.

Then

$$\sum_{k=1}^{\infty} f(k) \text{ converges}$$

$= f(k)\cdot 1 = f(k)\cdot(k+1-k)$

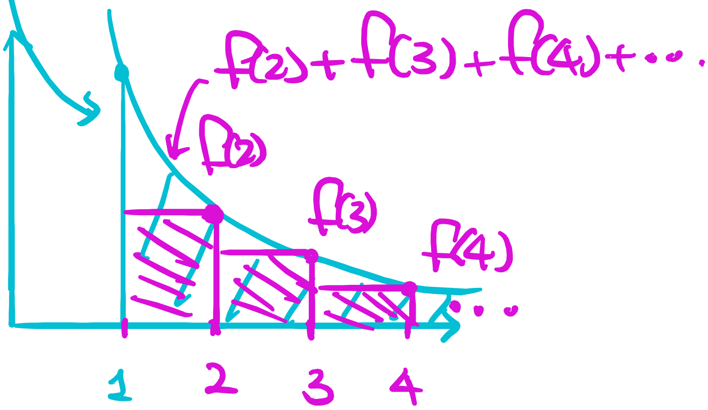
$$\Longleftrightarrow \int_1^{\infty} f(x)\,dx \text{ converges}$$




$$\text{Total area} = \sum_{k=1}^{\infty} f(k)$$

$$\geq \int_1^{\infty} f(x)\,dx$$

$$\geq \sum_{k=2}^{\infty} f(k)$$




Note

$f(k) \geq 0 \quad \forall k$

(ii) If $\int_1^{\infty} f(x)\,dx$ converges to $L^{<\infty}$,

then

then,
$$\left(\sum_{k=2}^{n} f(k)\right)_{n=2}^{\infty}$$

is increasing, bounded above by $L$.

$$\Rightarrow \sum_{k=2}^{\infty} f(k) \quad \text{Converges}$$

$$\Rightarrow \sum_{k=1}^{\infty} f(k) = f(1) + \sum_{k=2}^{\infty} f(k) \quad \text{Converges.}$$

(ii) If $\sum_{k=1}^{\infty} f(k)$ Converges to $M$,

then $\int_1^b f(x)\,dx \leq M \quad \forall b$

So $F(b) = \int_1^b \overset{>0}{\underline{f(x)}}\,dx$

is increasing. bounded above by $M$.

$$\Rightarrow \lim_{b\to\infty} F(b) = \int_1^{\infty} f(x)\,dx \quad \text{Converges.}$$

#

Exampl

Example

① Since

$$f(x) = \frac{1}{\sqrt{x}}$$

is decreasing, continuous, positive

$x^{-\frac{1}{2}} = 2(x^{\frac{1}{2}})'$

and

$$\int_1^\infty \frac{1}{\sqrt{x}}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{\sqrt{x}}\,dx$$

$$= \lim_{b\to\infty} 2\sqrt{x}\,\Big|_1^b$$

$$= \lim_{b\to\infty} 2(\sqrt{b} - 1)$$

diverges, we can conclude

that

$$\sum_{k=1}^{\infty}\frac{1}{\sqrt{k}}$$

diverges. #

② (p-series) Determine

$(p>0)$ $\infty$ 1

$$\sum_{k=1} \frac{1}{k^p}$$

Converges or diverges ?

Since

$$f(x) = \frac{1}{x^p}$$

is continuous, decreasing, positive

on $[1,\infty)$, and

$$x^{-p} = \begin{cases} (\ln|x|)', & p=1 \\ \left(\frac{x^{-p+1}}{-p+1}\right)', & p\neq 1 \end{cases}$$

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^p}\,dx$$

if $p\neq 1$

$$= \lim_{b\to\infty} \frac{x^{-p+1}}{-p+1} \Big|_1^b$$

$$= \lim_{b\to\infty} \left( \frac{b^{-p+1}}{-p+1} - \frac{1}{-p+1} \right)$$

$\to$ 0 if $p>1$

diverges if $p<1$

if $p=1$

$$= \lim_{b\to\infty} \ln|x| \Big|_1^b$$

$b \to \infty$

$$= \lim_{b\to\infty} \ln b \quad \text{diverges}$$

i.e. $$\int_1^\infty \frac{1}{x^p}\,dx \begin{cases} \text{converges if } p>1 \\ \text{diverges if } p\le 1 \end{cases}$$

we conclude that

$$\sum_{k=1}^{\infty} \frac{1}{k^p} \begin{cases} \text{converges if } p>1 \\ \text{diverges if } p\le 1 \end{cases}$$

#

<u>Thm</u> (Basic comparison test, Thm 12.3.6)

Let $a_k$ and $b_k$ be nonnegative.

Suppose

$$a_k \le b_k \quad \text{for large enough } k$$

Then

$\infty$

(i) if $\sum_{k=0}^{\infty} b_k$ converges, then

$$\sum_{k=0}^{\infty} a_k \text{ also converges;}$$

(ii) if $\sum_{k=0}^{\infty} a_k$ diverges, then

$$\sum_{k=0}^{\infty} b_k \text{ also diverges.}$$

Example

① $\displaystyle\sum_{k=1}^{\infty} \frac{1}{2k^3+1}$ converges because

$$\frac{1}{2k^3+1} < \frac{1}{k^3} \qquad \forall\, k \geq 1$$

and

$$\sum_{k=1}^{\infty} \frac{1}{k^3} \text{ converges} \qquad \#$$

② $\displaystyle\sum_{k=1}^{\infty} \frac{k^3}{k^5+5k^4+7}$ converges because

$$\frac{k^3}{\quad} < \frac{k^3}{\quad} = \frac{1}{\quad} \qquad \forall\, k \geq 1$$

$$k^5+5k^4+7 \qquad k^3 \qquad k^2 \qquad \forall k \geq 1$$

and

$$\sum_{k=1}^{\infty} \frac{1}{k^2} \text{ converges} \qquad \#$$

③ $$\sum_{k=1}^{\infty} \frac{1}{3k+1} \text{ diverges because}$$

$$\frac{1}{3k+1} \geq \frac{1}{3k+k} = \frac{1}{4} \cdot \frac{1}{k} \qquad \forall k \geq 1$$

and

$$\frac{1}{4} \sum_{k=1}^{\infty} \frac{1}{k} \text{ diverges} \qquad \#$$

<u>Thm</u> (Limit Comparison test, Thm 12.3.7)

Let $a_k, b_k > 0$. If

$$\lim_{k \to \infty} \frac{a_k}{b_k} \text{ exists}, \neq 0,$$

then

$$\sum_{k=1}^{\infty} a_k \text{ converges} \iff \sum_{k=1}^{\infty} b_k \text{ converges}$$

## Example

① $\displaystyle\sum_{k=1}^{\infty} \frac{1}{5^k - 3}$ converges because

$$\lim_{k\to\infty} \frac{\frac{1}{(5^k - 3)}}{\frac{1}{5^k}} = \lim_{k\to\infty} \frac{1}{1 - 3\cdot\frac{1}{5^k}} \quad \to 0$$

$$= 1 \neq 0$$

and

$\displaystyle\sum_{k=1}^{\infty} \frac{1}{5^k}$ converges #

② $\displaystyle\sum_{k=1}^{\infty} \frac{3k^2 + 2k + 1}{k^3 + 1}$ diverges because

$$\lim_{k\to\infty} \frac{\frac{3k^2 + 2k + 1}{k^3 + 1}}{\frac{1}{k}} = \lim_{k\to\infty} \frac{3k^3 + 2k^2 + k}{k^3 + 1} \quad \begin{matrix} \times \frac{1}{k^3} \\ \times \frac{1}{k^3} \end{matrix}$$

$$= \lim_{k\to\infty} \frac{3 + \frac{2}{k} + \frac{1}{k^2}}{1 + \frac{1}{k^3}} = 3 \neq 0$$

and

$\displaystyle\sum^{\infty} \frac{1}{\;}$ di...

$\sum_{k=1} \quad k$ converges #

③ $\displaystyle\sum_{k=1}^{\infty} \frac{2k+5}{\sqrt{k^6+3k^3}}$ Converges because

$$\lim_{k\to\infty} \frac{\frac{2k+5}{\sqrt{k^6+3k^3}}}{\frac{1}{k^2}} = \lim_{k\to\infty} \frac{2k^3+5k^2}{\sqrt{k^6+3k^3}} \begin{matrix} \times \frac{1}{k^3} \\ \times \frac{1}{k^3} \end{matrix}$$

$$= \lim_{k\to\infty} \frac{2+5\frac{1}{k}}{\sqrt{1+3\cdot\frac{1}{k^3}}} = 2 \neq 0$$

and

$\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges #

Thm (Root test, Thm 12.4.1)

Let $a_k \geq 0$. Suppose

$$\lim_{k\to\infty} \sqrt[k]{a_k} = \rho$$

Then

(i) if $\rho<1$, then $\sum_{k=1}^{\infty} a_k$ Converges;

(ii) if $\rho>1$, then $\sum_{k=1}^{\infty} a_k$ diverges;

(iii) if $\rho=1$, then $\sum^{\infty} a$ may

(iii) If $\rho = 1$, then $\sum_{k=1} a_k$ may

Converge or diverge (no conclusion)

e.g. Recall : $\lim_{k\to\infty} \sqrt[k]{k} = 1$.

① $\sum_{k=1}^{\infty} \frac{1}{k}$ : $\rho = \lim_{k\to\infty} \sqrt[k]{\frac{1}{k}} = \lim_{k\to\infty} \frac{1}{\sqrt[k]{k}} = 1$

diverges

② $\sum_{k=1}^{\infty} \frac{1}{k^2}$ : $\rho = \lim_{k\to\infty} \sqrt[k]{\frac{1}{k^2}} = \lim_{k\to\infty} \frac{1}{(\sqrt[k]{k})^2} = \frac{1}{1^2} = 1$

converges

pf

If $\rho = \lim_{k\to\infty} \sqrt[k]{a_k} < 1$, we can choose

$\mu$ so that

$$\lim_{k\to\infty} \sqrt[k]{a_k} = \rho < \boxed{\mu < 1}.$$

$\Rightarrow$ For $k$ sufficiently large, we have

$$\sqrt[k]{a_k} < \mu$$

$$\Leftrightarrow \boxed{a_k < \mu^k}$$

Since $\sum_{k=1}^{\infty} \mu^k$ Converges, by the basic comparison thm, $\sum_{k=1}^{\infty} a_k$ Converges #

<u>Example</u>

① Consider $\sum_{k=1}^{\infty} \frac{1}{(\ln k)^k}$.

$$\rho = \lim_{k\to\infty} \sqrt[k]{a_k} = \lim_{k\to\infty} \sqrt[k]{\frac{1}{(\ln k)^k}}$$

$$= \lim_{k\to\infty} \frac{1}{\ln k} = 0 < 1$$

Root test

$\Rightarrow$ $\sum_{k=1}^{\infty} \frac{1}{(\ln k)^k}$ Converges. #

② $\sum_{k=1}^{\infty} \frac{2^k}{k^3}$ :

$$\rho = \lim_{k\to\infty} \sqrt[k]{\frac{2^k}{k^3}} = \lim_{k\to\infty} \frac{2}{(\sqrt[k]{k})^3}$$

$\sqrt[k]{k} \to 1$

$$= 2 > 1$$

Root test

$\Rightarrow \sum_{k=1}^{\infty} \frac{2^k}{k^3}$ diverges #

Thm (Ratio test, Thm 12.4.2)

Let $a_k > 0$. Suppose

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lambda$$

(i) If $\lambda < 1$, then $\sum_{k=1}^{\infty} a_k$ converges

(ii) If $\lambda > 1$, then $\sum_{k=1}^{\infty} a_k$ diverges

(iii) If $\lambda = 1$, then NO conclusion.

e.g. Consider $\sum \frac{1}{k}$ and $\sum \frac{1}{k^2}$

$\lambda = \lim_{k\to\infty} \frac{k+1}{k} = 1$ $\qquad$ $\lim_{k\to\infty} \frac{(k+1)^2}{k^2} = 1^2 = 1$

Example

① $\sum_{k=0}^{\infty} \frac{1}{k!}$ :

$$\lambda = \lim_{k\to\infty} \frac{\frac{1}{(k+1)!}}{\frac{1}{k!}} = \lim_{k\to\infty} \frac{1}{k+1} = 0 < 1$$

Ratio Test

Ratio test

$\Rightarrow$ $\sum_{k=0}^{\infty} \frac{1}{k!}$ Converges.

② $\sum_{k=1}^{\infty} \frac{k}{10^k}$ :

$$\lambda = \lim_{k\to\infty} \frac{\frac{k+1}{10^{k+1}}}{\frac{k}{10^k}} = \lim_{k\to\infty} \underbrace{\frac{k+1}{k}}_{\to 1} \cdot \frac{1}{10}$$

$$= \frac{1}{10} < 1$$

Ratio test

$\Rightarrow$ $\sum_{k=1}^{\infty} \frac{k}{10^k}$ Converges. #