Calculus 3/12

Example

$$\int_0^1 \frac{1}{1-x}\,dx \quad \leftarrow = 0 \text{ at } x = 1$$

Recall $(x>0)$

$$(\ln x)' = \frac{1}{x}$$

$$= \lim_{c\to 1^-} \int_0^c \frac{1}{1-x}\,dx$$

$$\frac{1}{1-x} = (-\ln(1-x))'$$

$$(\ln(1-x))' = \frac{1}{1-x}\cdot(1-x)' = \frac{-1}{1-x}$$

$$= \lim_{c\to 1^-} (-\ln(1-x))\Big|_{x=0}^{c}$$

$$= \lim_{c\to 1^-} -\ln(1-c) + \underbrace{\ln(1-0)}_{=\ln 1 = 0}$$

$$= \lim_{c\to 1^-} \ln \frac{1}{1-c} \quad \text{diverges !!}$$

with $\frac{1}{1-c} \to \infty$ and $1-c \to 0$.

#

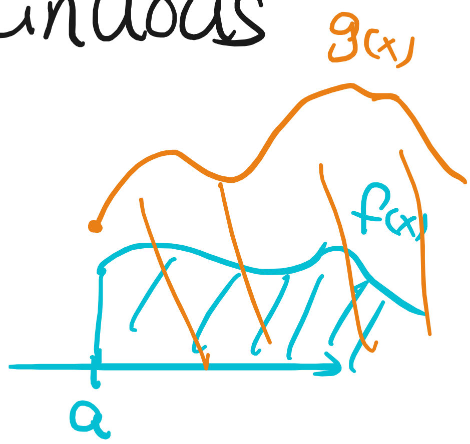
Q: Do we have an easier method

to determine the convergence of an improper integral?

<u>Thm</u> (11.7.2)

Let f and g be continuous on $[a, \infty)$ with

$$0 \le f(x) \le g(x)$$



for all $x \ge a$. Then

(i) If $\int_a^\infty g(x)\,dx$ converges, then $\int_a^\infty f(x)\,dx$ also converges.

(ii) If $\int_a^\infty f(x)\,dx$ diverges, then $\int_a^\infty g(x)\,dx$ also diverges.

Thm

Thm

If the positive functions $f(x)$ and $g(x)$ are continuous on $[a, \infty)$, and if

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = L, \quad 0 \lneq L \lneq \infty$$

then

$$\int_a^{\infty} f(x)\,dx \quad \text{and} \quad \int_a^{\infty} g(x)\,dx$$

either both converge or both diverge.

Recall (c.f. $\int_0^1 \frac{1}{x^p}\,dx = \begin{cases} \frac{1}{1-p}, & p<1 \\ \text{diverges}, & p \geq 1 \end{cases}$)

$$\int_1^{\infty} \frac{1}{\underbrace{x^p}_{>0}}\,dx = \begin{cases} \frac{1}{p-1}, & p>1 \\ \text{diverges}, & p \leq 1 \end{cases}$$

Example

$0 \leq \sin^2 x \leq 1$

① $\int_1^{\infty} \frac{\sin^2 x}{x^2}\,dx$ converges because

$\cdot$ 2

$$0 \leq \frac{\sin^{-1}x}{x^2} \leq \frac{1}{x^2} \text{ on } [1,\infty)$$

and $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ converges. #

② $\displaystyle\int_1^\infty \frac{1}{\sqrt{x^2-\frac{1}{2}}}\,dx$ diverges because

$$0 \leq \frac{1}{x} = \frac{1}{\sqrt{x^2}} \leq \frac{1}{\sqrt{x^2-\frac{1}{2}}} \text{ on } [1,\infty)$$

and $\displaystyle\int_1^\infty \frac{1}{x}\,dx$ diverges. #

③ $\displaystyle\int_1^\infty \frac{1}{1+x^2}\,dx$ converges because

$$\lim_{x\to\infty} \frac{\frac{1}{1+x^2}}{\frac{1}{x^2}} = \lim_{x\to\infty} \frac{\cancel{x^2}\,1}{\underbrace{\frac{1}{x^2}}_{\to 0}+\cancel{x^2}\,1} = 1 \begin{matrix}\neq 0\\ \neq \infty\end{matrix}$$

and

$$\int_1^\infty \frac{1}{x^2}\,dx \quad \text{converges} \qquad \#$$

④ $\displaystyle\int_1^\infty \frac{1-e^{-x}}{x}\,dx$ diverges because

$$\lim_{x\to\infty} \frac{\frac{1-e^{-x}}{x}}{\frac{1}{x}} = \lim_{x\to\infty} 1-e^{-x} = 1 \quad {}^{\neq 0}_{\neq \infty}$$

and $\displaystyle\int_1^\infty \frac{1}{x}\,dx$ diverges. $\#$

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§ Infinite series 級數

Consider

$$a_0 + a_1 + a_2 + \cdots$$

Basic question: do we really get

a number by this infinite sum:

More serious descriptions:

Let $(a_n)_{n=0}^{\infty}$ be a sequence.

Consider the seq. of partial sums:

$$\left(\sum_{k=0}^{n} a_k\right)_{n=0}^{\infty}$$

$$= a_0, (a_0+a_1), (a_0+a_1+a_2), \cdots$$

Def

We say the series $\sum_{k=0}^{\infty} a_k$ converges to L if the seq. $\left(\sum_{k=0}^{n} a_k\right)_{n=0}^{\infty}$ converges to L:

$$\sum_{k=0}^{\infty} a_k = \lim_{n\to\infty}\left(\sum_{k=0}^{n} a_k\right) = L.$$

We call L the sum of the series.

If $\left(\sum_{k=0}^{n} a_k\right)_{n=0}^{\infty}$ diverges, we say the series $\sum_{k=0}^{\infty} a_k$ diverges.

Example

① $$\sum_{k=0}^{\infty} \frac{1}{(k+1)(k+2)} = ?$$

$$= \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots$$

$$\sum_{k=0}^{n} \frac{1}{(k+1)(k+2)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{(n+1)(n+2)}$$

Note:

$$\frac{1}{(k+1)(k+2)} = \frac{1}{k+1} - \frac{1}{k+2}$$

$$\left(= \frac{(k+2)-(k+1)}{(k+1)(k+2)}\right)$$

So

$$\sum_{k=0}^{n} \frac{1}{(k+1)(k+2)} = \sum_{k=0}^{n} \frac{1}{k+1} - \frac{1}{k+2}$$

$$= \left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \cancel{\cdots}$$

$$\cancel{} + \left(\cancel{\frac{1}{n+1}} - \frac{1}{n+2} \right)$$

$$= 1 - \frac{1}{n+2}$$

So

$$\sum_{k=0}^{\infty} \frac{1}{(k+1)(k+2)} = \lim_{n\to\infty}\left(1 - \frac{1}{n+2}\right)$$

$$= 1 \quad \#$$

② $$\sum_{k=0}^{\infty} (-1)^k = ?$$

Consider

$$\sum_{k=0}^{n} (-1)^k = 1 - 1 + 1 - 1 + 1 - 1 \cdots + (-1)^n$$

$$= 1 \qquad \text{if } n = 0$$

$$= 1 - 1 = 0 \qquad n = 1$$

$$= 1 - 1 + 1 = 1 \qquad n = 2$$

$$= 1 - 1 + 1 - 1 = 0 \qquad n = 3$$

$$\vdots \qquad \vdots$$

$$\sum_{k=0}^{n}(-1)^k = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & n \text{ is odd.} \end{cases}$$

So

$$\sum_{k=0}^{\infty}(-1)^k \text{ diverges} \qquad \#$$

<u>Example</u> (Geometric series 等比級數)

Consider

$$\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots$$

$$\sum_{k=0}^{n} x^k = 1 + x + x^2 + \cdots + x^n$$

(<u>Note</u>

$$1 - x^{n+1} = (1-x)(1 + x + x^2 + \cdots + x^n)$$)

$x^{n+1} \to 0$ if $|x| < 1$

diverges if $|x| > 1$

$$= \begin{cases} \dfrac{1 - x^{n+1}}{1 - x} & \text{if } x \neq 1 \end{cases}$$

$($ $\underline{n+1} \to \infty$ as $n \to \infty$ if $x = 1$

So

$$\sum_{k=0}^{\infty} x^k \begin{cases} = \dfrac{1}{1-x} & \text{if } |x| < 1 \\ \text{diverges} & \text{if } |x| > 1 \\ \text{diverges} & x = -1 \\ \text{diverges.} & x = 1 \end{cases}$$

$$\sum_{k=0}^{\infty} x^k \begin{cases} = \dfrac{1}{1-x} & \text{if } |x| < \cancel{0}\ 1 \\ \text{diverges} & \text{if } |x| \geq 1 \end{cases}$$

#

Remark

Let p be a positive integer.

$\sum_{k=0}^{\infty} a_k$ converges to L

$\Leftrightarrow$ $= (a_1 + a_2 + \dots a \ ) + (a + a + \dots$ $)$

$(a_0 + a_1 + \cdots + a_{p-1}) + (a_p + a_{p+1} \cdots)$

$\sum_{k=p}^{\infty} a_k$ converges to

$$L - \left(\sum_{k=0}^{p-1} a_k\right)$$

Thm (Thm 12.2.4, Thm 12.2.5)

Let $(a_n)_{n=0}^{\infty}$, $(b_n)_{n=0}^{\infty}$ be seq.

α, β be numbers

(i) If $\sum_{k=0}^{\infty} a_k$ converges to L and

$\sum_{k=0}^{\infty} b_k$ converges to M

then

$$\sum_{k=0}^{\infty} (\alpha a_k + \beta b_k) \left(= \sum_{k=0}^{\infty} \alpha a_k + \sum_{k=0}^{\infty} \beta b_k = \alpha \sum_{k=0}^{\infty} a_k + \beta \sum_{k=0}^{\infty} b_k \right)$$

converges to

$$\alpha L + \beta M$$

Warning:

$$\sum_{k=0}^{\infty} a_k b_k \neq L \cdot M$$

(ii) If $\sum_{k=0}^{\infty} a_k$ converges, then

$$\lim_{k\to\infty} a_k = 0.$$

Equivalently, if $\lim_{k\to\infty} a_k \neq 0$, then $\sum_{k=0}^{\infty} a_k$ diverges.

e.g. $\sum_{k=0}^{\infty} k$ diverges. since $k \not\to 0$ as $k\to\infty$

$\sum_{k=0}^{\infty} (-1)^k$ diverges since $(-1)^k \not\to 0$ as $k\to\infty$

pf of (ii)

Assume $\sum_{k=0}^{\infty} a_k$ converges. That is, the seq.

$$\left(\sum_{k=0}^{n} a_k\right)_{n=0}^{\infty} \qquad \lim_{n\to\infty}\left(\sum_{k=0}^{n} a_k\right)$$

Converges to a number L.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\begin{pmatrix}(a_0+a_1+\cdots+a_{n-1}+a_n) \\ -(a_0+a_1+\cdots+a_{n-1})\end{pmatrix}$$

$$= \lim_{n\to\infty}\left(\sum_{k=0}^{n} a_k - \sum_{k=0}^{n-1} a_k\right)$$

$\because \left(\sum_{k=0}^{n} a_k\right)_{n=0}^{\infty}$ converges

and $\left(\sum_{k=0}^{n-1} a_k\right)$ converges

$$= \lim_{n\to\infty}\sum_{k=0}^{n} a_k - \lim_{n\to\infty}\sum_{k=0}^{n-1} a_k$$

$\Leftrightarrow n-1\to\infty$

Recall: if (b_n) (c_n) converge, then

$$\lim_{n\to\infty}(b_n - c_n) = \lim_{n\to\infty} b_n - \lim_{n\to\infty} c_n$$

$$= L - L = 0$$

#

$\sum\left(\frac{1}{2}\right)^k$ $\sum\left(\frac{1}{3}\right)^k$ converge

Example

① $\sum^{\infty}\left(\left(\frac{1}{2}\right)^k - \left(\frac{1}{3}\right)^k\right)$

$$= \sum_{k=0} \left(\left(\frac{1}{2}\right) + 2 \cdot \left(\frac{1}{3}\right) \right)$$

$$= \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k + 2 \cdot \sum_{k=0}^{\infty} \left(\frac{1}{3}\right)^k$$

$$= \frac{1}{1-\frac{1}{2}} + 2 \cdot \frac{1}{1-\frac{1}{3}}$$

$$= 2 + 2 \cdot \frac{3}{2} = 5 \quad \#$$

② $\sum_{k=0}^{\infty} \frac{k}{k+1}$ diverges because

$$\lim_{k\to\infty} \frac{k}{k+1} = \lim_{k\to\infty} \frac{1}{1+\frac{1}{k}} = 1 \neq 0$$

$\#$

③ $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} = ?$

$$\sum_{k=1}^{n} \frac{1}{\sqrt{k}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}}$$

$$\geq \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + \cdots + \frac{1}{\sqrt{n}}$$

$$= n \cdot \frac{1}{\sqrt{n}} = \sqrt{n} \longrightarrow \infty$$

as $n \to \infty$

So $\left(\sum_{k=1}^{n} \frac{1}{\sqrt{k}} \right)_{n=1}^{\infty}$ is unbounded

$\Rightarrow \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ diverges. #