Calculus 3/7

<u>Example</u> (Algorithms of searching)

Q: Find a certain word in a dictionary
(Design algorithms)

Assume there are n words in the dictionary.

<u>Algorithm 1</u>: Check each word one by one.

<u>Worst case</u>: wanted word is the last one

— Need n steps to find the word.

Time complexity of Algorithm 1 $= O(n)$.

Algorithm 2 (binary search algorithm)

Know: the words in dictionary are in the alphabet order.

Steps: ① Open a page (divide the dictionary into 2 parts)

② Check the word is in which part.

③ Repeat ① ~ ②

→ k steps can cover 2^k words

Worst case: Need $\log_2 n$ steps to find the word.

⇒ Time Complexity = $O(\log n)$

Conclution

Since

$$\lim_{n\to\infty} \frac{\log n}{n} = 0,$$

Algorithm 2 is faster (for large n)

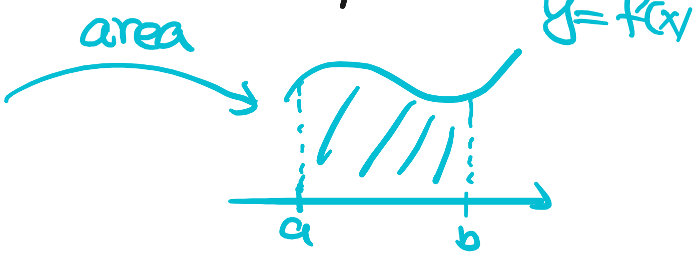
better in general.

§ Improper integrals (瑕積分)

Recall:

In Fall semester, we mostly consider

$$\int_a^b f(x)\,dx$$



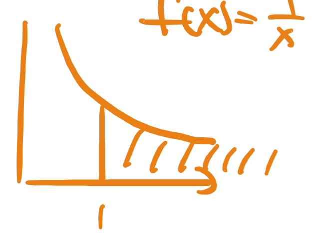
for continuous functions f on $[a,b]$

not good

$f(x) = \frac{1}{x}$ is NOT continuous on $[0,1]$

improper

integrals —



$\int_1^\infty f(x)\,dx = ?$

Recall

Assume $a,b \in \mathbb{R}$, $a,b \neq \infty$, f is continuous on $[a,b]$.

Then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

if $F'(x) = f(x)$ on $[a,b]$

Def (§11.7)

Integrals with infinite upper/lower bounds are called improper integrals (of Type I).

① If $f(x)$ is continuous on $[a,\infty)$, then

$$\int_a^\infty f(x)dx \overset{\text{def.}}{=} \lim_{b\to\infty}\int_a^b f(x)\,dx.$$

② If $f(x)$ is continuous on $(-\infty, b]$,
then

$$\int_{-\infty}^{b} f(x)dx \overset{\text{def.}}{=} \lim_{a \to -\infty} \int_{a}^{b} f(x)\, dx$$

③ If $f(x)$ is continuous on $(-\infty, \infty)$,
then

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{c} f(x)dx + \int_{c}^{\infty} f(x)\, dx$$

where c is any real number.

In each case, if the limit exists and finite, we say the improper integral converges and the limit is the value of the improper integral.
If the limit does not exist, we say the improper integral diverges.

Example

Example

① $\displaystyle\int_1^\infty \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty} \boxed{\int_1^b \frac{\ln x}{x^2}\,dx}$

Recall (integration by parts)

$\displaystyle\int_a^b u'(x)\cdot v(x)\,dx$

$\displaystyle= u(x)\cdot v(x)\Big|_{x=a}^{b} - \int_a^b u(x)\cdot v'(x)\,dx$

$\displaystyle\int_1^b \underbrace{\frac{1}{x^2}}_{u'}\cdot \overbrace{\ln x}^{v}\,dx$

$\displaystyle= \left(-\frac{1}{x}\right)\cdot \ln x\Big|_1^b - \int_1^b \left(-\frac{1}{x}\right)\cdot\frac{1}{x}\,dx$

$u'(x) = \frac{1}{x^2} = x^{-2}$

$u(x) = -x^{-1} = -\frac{1}{x}$

$v(x) = \ln x,$

$v'(x) = \frac{1}{x}$

$\ln x = \int_1^x \frac{1}{t}\,dt$

$\displaystyle= -\frac{1}{b}\cdot \ln b - \cancel{\left(-\frac{1}{1}\cdot \overset{0}{\overset{\parallel}{\ln 1}}\right)}$

$\displaystyle+ \int_1^b \boxed{x^{-2}}\,dx$

Recall:

$(x^a)' = a\cdot x^{a-1}$

$\Rightarrow (-x^{-1})' = x^{-2}$

$\displaystyle= -\frac{\ln b}{b} + \left(-\frac{1}{x}\right)\Big|_1^b$

$\displaystyle= -\frac{\ln b}{b} - \left(\frac{1}{b} - \frac{1}{1}\right)$

$$= -\frac{\ln b}{b} - \frac{1}{b} + 1$$

So
$$\int_1^\infty \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty}\left(-\cancel{\frac{\ln b}{b}}^{\,0} - \cancel{\frac{1}{b}}^{\,0} + 1\right)$$

$$= 1 \qquad \#$$

② $$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_0^\infty \frac{dx}{1+x^2}$$

Recall

$$(\tan^{-1}x)' = \frac{1}{1+x^2}$$

$$\Rightarrow \int_a^b \frac{1}{1+x^2}\,dx = \tan^{-1}b - \tan^{-1}a$$

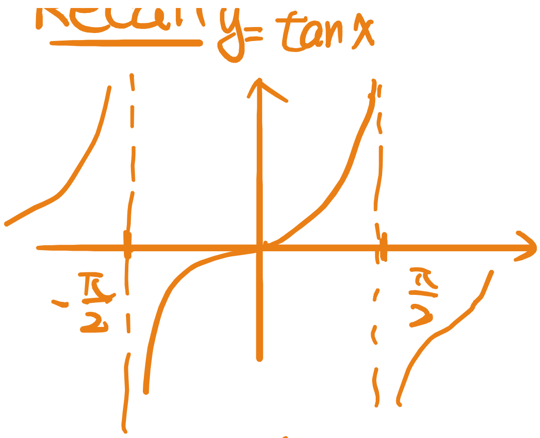
(i) $$\int_0^\infty \frac{1}{1+x^2}\,dx = \lim_{b\to\infty}\int_0^b \frac{1}{1+x^2}\,dx$$

$$= \lim_{b\to\infty}\left(\tan^{-1}b - \overbrace{\cancel{\tan^{-1}0}}^{=0}\right)$$

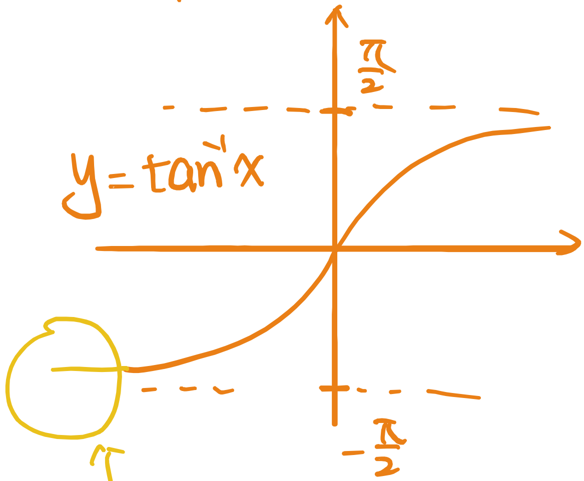
Recall..

$$= \lim_{b \to \infty} \tan^{-1} b$$

$$= \frac{\pi}{2}$$



(ii) $$\int_{-\infty}^{0} \frac{1}{1+x^2}\,dx$$



$$= \lim_{a \to -\infty} \int_{a}^{0} \frac{1}{1+x^2}\,dx$$

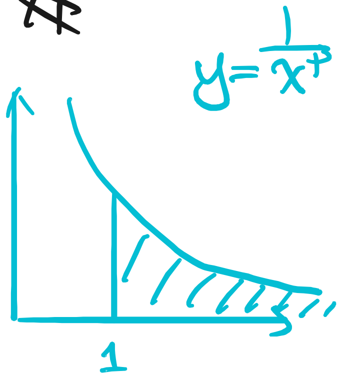
$$= \lim_{a \to -\infty} \left(\underbrace{\cancel{\tan^{-1} 0}}_{} ^{=0} - \tan^{-1} a \right)$$

$$= -\left(-\frac{\pi}{2}\right) = \frac{\pi}{2}$$

So $$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{\infty} \frac{dx}{1+x^2}$$

$$= \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

$$= \frac{\pi}{2} + \frac{\pi}{2} = \pi \quad \#$$



③

$$\int_1^\infty \frac{1}{x^p}\,dx$$

$$= \lim_{b\to\infty} \int_1^b \frac{1}{x^p}\,dx$$

$$= x^{-p} = \begin{cases} \left(\dfrac{x^{-p+1}}{-p+1}\right)' & \text{if } p \neq 1 \\ (\ln x)' & \text{if } p = 1 \end{cases}$$

$$= \lim_{b\to\infty} \left(\begin{cases} \left.\dfrac{x^{-p+1}}{-p+1}\right|_1^b & \text{if } p \neq 1 \\ \left.\ln x\right|_1^b & \text{if } p = 1 \end{cases} \right)$$

$$b^{-p+1} = \frac{1}{b^{p-1}} \begin{cases} \to 0 & \text{if } p > 1 \\ \to \infty & \text{if } p < 1 \end{cases}$$

$$= \lim_{b\to\infty} \left(\frac{b^{-p+1}}{-p+1} - \frac{1}{-p+1} \right) \quad \text{if } p \neq 1$$

$$= \begin{cases} \dfrac{1}{-1+p} & \text{if } p > 1 \\ \text{diverges} & \text{if } p < 1 \end{cases}$$

$$= \lim_{b\to\infty} \left(\underbrace{\ln b}_{\to\infty \text{ as } b\to\infty} - \overset{=0}{\cancel{\ln 1}} \right) \quad \text{if } p = 1$$

diverges

So

$$\int_1^\infty \frac{1}{x^p}\,dx = \begin{cases} \dfrac{1}{p-1}, & \text{if } p > 1 \\ \text{diverges}, & \text{if } p \le 1 \end{cases}$$

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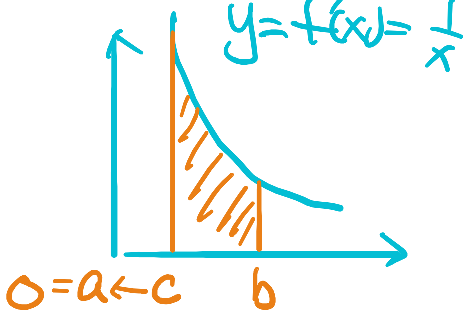
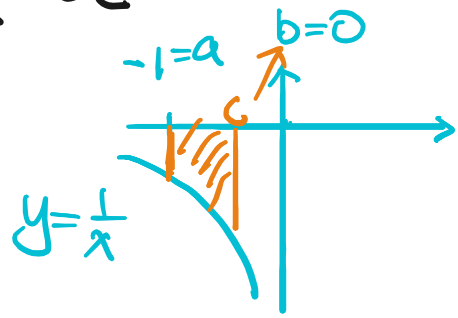
<u>Def</u>

Integrals of functions that become infinite at a point within the integration interval are called improper integrals (of called improper integrals (or Type II)

① If $f(x)$ is continuous on $(a, b]$ and $\lim_{x\to a} f(x)$ doesn't exist, then

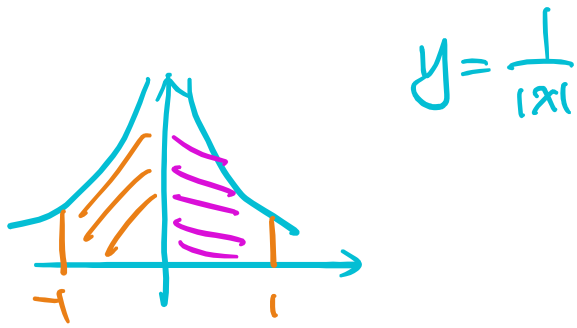
$$\int_a^b f(x)\,dx = \lim_{c\to a^+} \int_c^b f(x)\,dx$$





② If f is continuous $[a, b)$ and discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{c\to b^-} \int_a^c f(x)\,dx$$



③ If f is distinuous at $c \in (a,b)$

and continuous on $[a,c) \cup (c,b]$

then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

In each case, if the limit exists (finite), we say the improper integral converges and the limit is the value of the improper integral.

If the limit does not exist, we say the improper integral diverges.

Example

Note: $\frac{1}{x^p} \to \infty$ as $x \to 0$

$$\int_0^1 \frac{1}{x^p}\,dx$$

$p \neq 1$

$-p+1$

$\ln x \qquad p=1$

$$= \lim_{c\to 0^+} \int_c^1 \frac{1}{x^p}\,dx$$

$\to 0$ if $-p+1>0$ $\Leftrightarrow$ $p<1$

$\to \infty$ if $-p+1<0$ $\Leftrightarrow$ $p>1$

if $p\neq 1$

$$= \lim_{c\to 0^+}\left(\frac{1}{-p+1} - \frac{c^{-p+1}}{-p+1}\right)$$

$$= \begin{cases} \dfrac{1}{1-p} & \text{if } p<1 \\ \text{diverges} & \text{if } p>1 \end{cases}$$

if $p=1$

$$= \lim_{c\to 0^+}(-\ln c) \quad \text{diverges}$$

So

$$\int_0^1 \frac{1}{x^p}\,dx = \begin{cases} \dfrac{1}{1-p} & \text{if } p<1 \\ \text{diverges} & \text{if } p\geq 1 \end{cases}$$

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