# Calculus 3/5

## Recall

We considered

$$\lim_{x \to c} f(x) \quad \text{and} \quad \lim_{x \to \infty} f(x)$$

in Fall. They are related to seq. in the following ways:

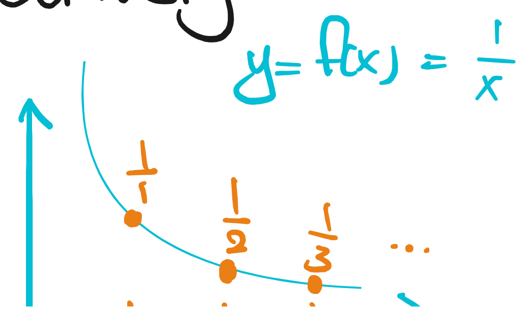
## Prop

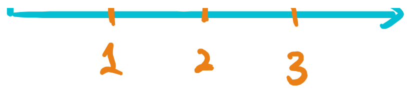
Let $f$ be a function, $c$ a number.

① If $\lim_{x \to \infty} f(x) = L$, then the seq.

$$(a_n = f(n))_{n=1}^{\infty} : f(1), f(2), f(3), \dots$$

also converges to $L$.



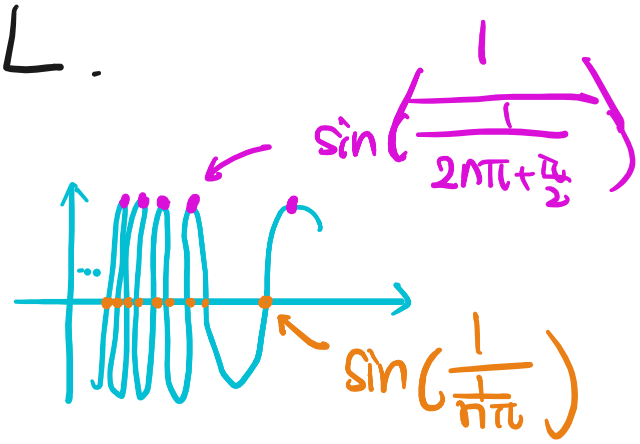



② If $\lim_{x \to c} f(x) = L$ and if $\lim_{n\to\infty} c_n = c$,

then the seq.

$$\left(b_n = f(c_n)\right)_{n=1}^{\infty} : f(c_1),\ f(c_2),\ f(c_3), \dots$$

converges to $L$.




<u>Application:</u>

Consider

$$f(x) = \sin\left(\frac{1}{x}\right), \quad x > 0.$$

The limit $\lim_{x\to 0} \sin\left(\frac{1}{x}\right)$ does not exist

because for sequences

$$\frac{1}{n\pi} \to 0, \qquad \frac{1}{2n\pi + \frac{\pi}{2}} \to 0$$

as $n \to \infty$

we have

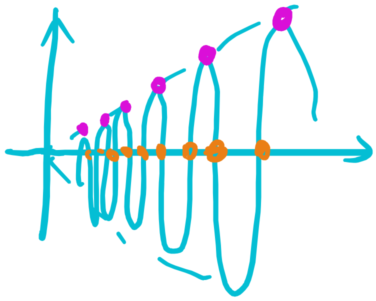
$$\sin\left(\frac{1}{\frac{1}{n\pi}}\right) = \sin(n\pi) = 0 \to 0$$

$$\sin\left(\frac{1}{2n\pi+\frac{\pi}{2}}\right) = \sin\left(2n\pi+\frac{\pi}{2}\right) = 1 \to 1$$

## Convergent case:

$f(x) = x\sin\frac{1}{x}$

$\to 0$ as $x \to 0$



# Some important limits

Prop (§11.4)

① $\lim_{n\to\infty} \frac{1}{n} = 0$

Consider $f(x) = a^x$

— continuous at $x=0$

② $\lim_{n\to\infty} a^{\frac{1}{n}} = a^0 = 1$ (Assume $a>0$)

③ $\lim_{n\to\infty} a^n = 0$ if $|a|<1$

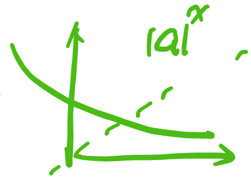
pf

$\max\{1, \log_{|a|}\frac{\varepsilon}{2}\}$  $\log_{|a|}\varepsilon$

Given $\varepsilon>0$, take $K = \cancel{\log_{|a|}\left(\max\{\tfrac{1}{2}, \tfrac{\varepsilon}{2}\}\right)} > 0$

decreasing

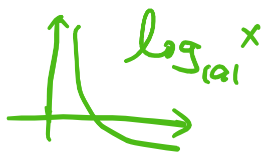
$|a|<1$



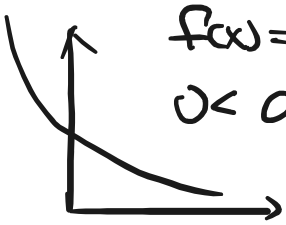

$\Rightarrow \forall n \geq K,$

$$|a^n - 0| = |a|^n \leq |a|^K$$

$$< |a|^{\log_{|a|}\varepsilon} = \varepsilon$$



Another explanation:



$f(x) = a^x \to 0$ as $x \to \infty$

$0 < a < 1$

$\Downarrow$

$a^n \to 0$ as $n \to \infty$

④ $\lim_{n\to\infty} \frac{1}{n^a} = 0$ if $a > 0$

because $f(x) = x^a$ is (right) continuous at 0

e.g. $a = \frac{1}{2}$ $f(x) = \sqrt{x}$

$a = 3$ $f(x) = x^3$

and $\frac{1}{n} \to 0$ from right

$\Rightarrow \lim_{n\to\infty} f\left(\frac{1}{n}\right) = f(0) = 0$

⑤ $\lim_{n\to\infty} \frac{a^n}{n!} = 0$ for any real number $a$.

Notation:

$[|a|]$ = 小於 (等於) $|a|$ 的最大整數

e.g. $[1.5] = 1$, $[2.1] = 2$

Pf

For $n > |a|$,

$\left| \frac{a^n}{n!} - 0 \right| = \frac{|a|^n}{\ldots}$

$$n! \qquad\qquad n!$$

$$= \underbrace{\left(\frac{|a|}{1}\cdot\frac{|a|}{2}\cdots\frac{|a|}{[|a|]}\right)}_{\substack{\text{independent of } n,\\ \text{Call it } M}}\cdot\underbrace{\left(\overset{<1}{\underbrace{\frac{|a|}{[|a|]+1}}}\cdots\overset{<1}{\underbrace{\frac{|a|}{n}}}\right)}_{< \frac{|a|}{n}}$$

So

$$0\le\left|\frac{a^n}{n!}-0\right| < M\cdot\frac{|a|}{n}\to 0 \text{ as } n\to\infty$$

$$\forall n>|a|$$

By Pinching Thm,

$$\lim_{n\to\infty}\left|\frac{a^n}{n!}-0\right| = 0$$

$$\Leftrightarrow \lim_{n\to\infty}\frac{a^n}{n!} = 0 \qquad \#$$

⑥ $$\lim_{n\to\infty}\frac{\ln n}{n} = 0$$

pf

Consider $f(x)=\dfrac{\ln x}{x}$.

By L'Hôpital's rule,

$$\lim \ \overset{\to\infty}{(\ln x)} \qquad\qquad \lim \ (\ln x)'$$

$$\lim_{x\to\infty}\frac{\ln x}{\underset{\to\infty}{x}} = \lim_{x\to\infty}\frac{(\ln x)'}{(x)'}$$

$$= \lim_{x\to\infty}\frac{\frac{1}{x}}{1} = 0$$

So $\displaystyle\lim_{n\to\infty}\frac{\ln n}{n} = 0$ #

⑦ $\displaystyle\lim_{n\to\infty} n^{\frac{1}{n}} = 1$

pf

Consider $f(x) = x^{\frac{1}{x}}$, $x > 0$

$$\lim_{x\to\infty} x^{\frac{1}{x}} = \lim_{x\to\infty} e^{\ln x^{\frac{1}{x}}}$$

$$= \lim_{x\to\infty} e^{\frac{\ln x}{x}}$$

$$= e^{\underbrace{\lim_{x\to\infty}\frac{\ln x}{x}}_{=0}} = e^0 = 1$$

So (take $x = n$)

$$\lim_{n\to\infty} n^{\frac{1}{n}} = 1 \qquad \#$$

<u>Thm</u> (§11.4)

$$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$$

More generally,

$$\lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n = e^x$$

pf

For $x = 0$,

$$\lim_{n\to\infty} \left(1+\frac{0}{n}\right)^n = 1 = e^0.$$

For $x \neq 0$, Consider

$$\ln\left(\left(1+\frac{x}{n}\right)^n\right) = n\ln\left(1+\frac{x}{n}\right)$$

$$= \underset{\sim}{x} \cdot \frac{\ln(1+ x/n) - \ln 1}{x/n} \to 1$$

$h = \frac{x}{n} \to 0$

Recall:

$$(\ln x)'\Big|_{x=1} = \lim_{h \to 0} \frac{\ln(1+h) - \ln 1}{h}$$

$$= \frac{1}{x}\Big|_{x=1} = \underline{\underline{1}}$$

So

$$\lim_{n\to\infty} \ln\left(\left(1+\frac{x}{n}\right)^n\right) = x$$

$$\Rightarrow \quad \underset{\sim\sim\sim}{e^x =} e^{\lim_{n\to\infty} \ln\left(\left(1+\frac{x}{n}\right)^n\right)}$$

$$= \lim_{n\to\infty} e^{\ln\left(\left(1+\frac{x}{n}\right)^n\right)}$$

$$= \lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n \qquad \#$$

Relative rate of growth

We will state definitions / thms by " $\lim_{x\to\infty} f(x)$ ". You can replace it by " $\lim_{n\to\infty} a_n$ ".

<u>Def</u>

Let $f(x)$ and $g(x)$ be positive for $x$ sufficiently large.

(i) We say " $f$ grows faster than $g$ " as $x \to \infty$ if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty,$$

or equivalently, if

$$\lim_{x\to\infty} \frac{g(x)}{f(x)} = 0$$

We also say " $g$ grows slower

than f".

② We say "f and g grow the same rate" as $x \to \infty$ if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$$

where $L \neq \infty$, $L \neq 0$.

<u>Example</u>

① Since

$$\lim_{x\to\infty} \frac{3x^2+x+5}{x^2+1} = 3 \neq 0$$

the functions

$3x^2+x+5 = O(x^2+1)$
$x^2+1 = O(3x^2+x+5)$

$3x^2+x+5$ and $x^2+1$

grow at the same rate.

② Since

$$\lim \frac{100x^4+1}{\ldots} = \ldots$$

$$\lim_{x\to\infty} \quad x^5+x^4+x^3$$

the function $\quad 100x^4+1 = o(x^5+x^4+x^3)$

$x^5+x^4+x^3$

grows faster than $100x^4+1$

③ $e^x$ grows faster than $x^2$ as $x\to\infty$

$x^2 = o(e^x)$

since

L'Hôpital

$$\lim_{x\to\infty} \frac{x^2}{e^x} = \lim_{x\to\infty} \frac{2x}{e^x}$$

$$= \lim_{x\to\infty} \frac{2}{e^x} = 0$$

④ $3^x$ grows faster than $2^x$

$2^x = o(3^x)$ as $x\to\infty$

since

$$\lim_{x\to\infty} \frac{2^x}{3^x} = \lim_{x\to\infty} \left(\frac{2}{3}\right)^x = 0$$

⑤ $x^2$ grows faster than $\ln x$

$$\ln x = o(x^2) \quad \text{as } x\to\infty$$

since

$$\lim_{x\to\infty}\frac{\ln x}{x^2} = \lim_{x\to\infty}\frac{\frac{1}{x}}{2x} = 0$$

⑥ $x^{\frac{1}{n}}$ grows faster than $\ln x$

$$\ln x = o(x^{\frac{1}{n}})$$

Since

$$\lim_{x\to\infty}\frac{\ln x}{x^{\frac{1}{n}}} = \lim_{x\to\infty}\frac{\cancel{\frac{1}{x}}}{\frac{1}{n}\cdot \cancel{x^{\frac{1}{n}-1}}} \quad n\cdot\frac{1}{x^{\frac{1}{n}}}$$

$$= 0$$

<u>Def</u>

① We say "f is of smaller order than g" as $x\to\infty$ if

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = 0$$

denoted by $f = o(g)$

"f' is little-oh of' g"

② We say "f is of at most the order of g" as $x \to \infty$ if $\exists\, M > 0$ s.t.

$$\frac{f(x)}{g(x)} \leq M$$

$\left(\Rightarrow \lim_{x\to\infty} \frac{f(x)}{g(x)} \neq \infty\right)$

for x sufficiently large, denoted by

$$f = O(g)$$

" f is big-oh of g"