# Calculus 2/29

<u>Problem:</u>

In general, it is difficult to determine whether a seq. converges or diverges by the definition ($\varepsilon \cdots$).

We need a few thms to help us determine the converge of a seq.

<u>Thm</u>

① Every convergent seq. is bounded.

② Every unbounded seq. is divergent.

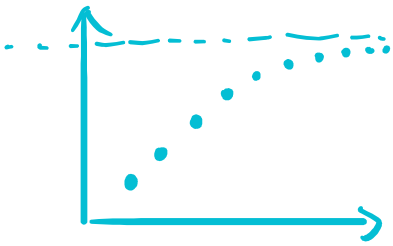
eg.

$a_n = n$ , $n = 1, 2, 3, \cdots$

is unbounded, so it is divergent.

③ A bounded above increasing seq. is convergent.

④ A bounded below decreasing seq. is convergent.




<u>Example</u>

① $\left(a_n = \frac{1}{n}\right)_{n=1}^{\infty}$ is decreasing, bounded below by 0

$\Rightarrow$ it is convergent.

② $\left(a_n = \frac{n}{n+1}\right)_{n=1}$

$= 1 - \frac{1}{n+1}$

is increasing, bounded above by 1.


$\Rightarrow$ it is convergent.

③ $\left(a_n = \frac{n}{e^n}\right)_{n=1}^{\infty}$

Consider

$$f(x) = \frac{x}{e^x}$$

$\left( a_n = f(n) \quad n = \underbrace{1, 2, 3, \ldots}_{\geq 1} \right)$

Since

$$f'(x) = e^{-x} + xe^{-x}(-1)$$
$$= (1-x)e^{-x} \leq 0$$

$\forall x \geq 1$, we know $f$ is

decreasing.

$\Rightarrow \quad a_n = f(n) \geq f(n+1) = a_{n+1}$

$\Rightarrow \quad \left(a_n = \frac{n}{e^n}\right)_{n=1}^{\infty}$ is decreasing

$>0$

bounded below by 0

$\Rightarrow$ it is convergent. #

Def

子數列

A subsequence of a sequence $(a_n)_{n=1}^{\infty}$ is a seq. of the form

$$(a_{n_k})_{k=1}^{\infty} = a_{n_1}, a_{n_2}, \cdots$$

where $(n_k)_{k=1}^{\infty}$ is a strictly increasing seq. of positive integers.

Example

Consider

$$\left(a_n = (-1)^{n+1}\right)_{n=1}^{\infty} =$$

$$1, -1, 1, -1, 1, -1, \dots$$

Then

(i) $1, 1, 1, 1, \dots$ $\quad (n_k = 2k-1)$

(ii) $-1, -1, -1, -1, \dots$ $\quad (n_k = 2k)$

(iii) $1, -1, 1, 1, 1, \dots$ $\quad \left(n_k = \begin{cases} k & , k=1,2,3 \\ 2k-3 & , k \geq 4 \end{cases}\right)$

are subsequences of $(a_n)_{n=1}^{\infty}$.

Prop = Proposition $\approx$ Theorem

If a seq. converges to $L$, then all its subsequences converge to $L$.

## Example

The seq. $\left(a_n = (-1)^{n+1}\right)_{n=1}^{\infty}$ diverges

because it has

a subseq. $\longrightarrow 1$

$\neq$

a subseq. $\longrightarrow -1$.

**Thm** (Thm 11.3.7)

Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be convergent, and $\alpha \in \mathbb{R}$.

real numbers

in

① $$\lim_{n\to\infty}(a_n+b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$

② $$\lim_{n\to\infty}(\alpha \cdot a_n) = \alpha \cdot \lim_{n\to\infty} a_n$$

③ $$\lim_{n\to\infty}(a_n \cdot b_n) = \left(\lim_{n\to\infty} a_n\right)\cdot\left(\lim_{n\to\infty} b_n\right)$$

Convergent

④ If ($b_n \neq 0$, for $n$ large enough) $\lim_{n\to\infty} b_n \neq 0$, (*)
then

(*)

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$$

**Example**

Compute the limits.

(Recall: $\lim_{n\to\infty}\frac{1}{n} = 0$ )

① $$\lim_{n\to\infty}\frac{3n^4 - 2n^2 + 1}{n^5 - 3n^3} \quad \begin{matrix}\times \frac{1}{n^5} \\ \times \frac{1}{n^5}\end{matrix}$$

$$= \lim_{n\to\infty}\frac{3\frac{n^4}{n^5} - 2\frac{n^2}{n^5} + \frac{1}{n^5}}{\frac{n^5}{n^5} - 3\frac{n^3}{n^5}}$$

$$= \lim_{n\to\infty} \frac{3\cdot\frac{1}{n} - 2\left(\frac{1}{n}\right)^3 + \left(\frac{1}{n}\right)^5}{1 - 3\left(\frac{1}{n}\right)^2}$$

$\to 1 - 3\cdot 0^2 = 1 \neq 0$

④ in Thm

$$= \frac{\lim_{n\to\infty}\left(3\cdot\frac{1}{n} - 2\left(\frac{1}{n}\right)^3 + \left(\frac{1}{n}\right)^5\right)}{\lim_{n\to\infty}\left(1 - 3\left(\frac{1}{n}\right)^2\right)}$$

①,②,③ in Thm

$$= \frac{3\cdot\big(\underbrace{\lim_{n\to\infty}\tfrac{1}{n}}_{=0}\big) - 2\big(\underbrace{\lim_{n\to\infty}\tfrac{1}{n}}_{=0}\big)^3 + \big(\underbrace{\lim_{n\to\infty}\tfrac{1}{n}}_{=0}\big)^5}{1 - 3\big(\underbrace{\lim_{n\to\infty}\tfrac{1}{n}}_{=0}\big)^2}$$

$$= \frac{0}{1} = \underline{\underline{0}} \quad \#$$

② $$\lim_{n\to\infty} \frac{1 - 4n^7}{1n^7 + 12n} \quad \begin{matrix} \times \frac{1}{n^7} \\ \times \frac{1}{n^7} \end{matrix}$$

$$= \lim \frac{\left(\frac{1}{n}\right)^7 - 4}{1 + 12\left(\frac{1}{n}\right)^6} = \frac{-4}{1} = -4$$

$$\lim_{n\to\infty} \quad 1 + 12\left(\tfrac{1}{n}\right) \to 0$$

③ $$\lim_{n\to\infty} \frac{n^4 - 3n^2 + n + 2}{n^3 - 7n} \; \frac{\times \frac{1}{n^4}}{\times \frac{1}{n^4}}$$ doesn't exist

$a_n$ =

$$= \lim_{n\to\infty} \frac{1 - 3\frac{1}{n^2} + \frac{1}{n^3} + \frac{2}{n^4}}{\underbrace{\frac{1}{n} - 7\frac{1}{n^3}}} \to 0$$

Since

$$a_n = \frac{1}{\frac{1}{n} - 7\frac{1}{n^3}} + \frac{-3\frac{1}{n} + \frac{1}{n^2} + \frac{2}{n^3}}{1 - 7\frac{1}{n^2}}$$

$\to 0$, $\to 1$

Converges to 0
In particular, bounded.

$$\frac{n}{\underbrace{1 - 7\frac{1}{n^2}}} \geq 1 - \frac{7}{9} = \frac{2}{9}$$

for $n \geq 3$

$$\geq \frac{2}{9} n \quad \forall n \geq 3$$

unbounded

correction:

$0 < 1 - 7\frac{1}{n^2} < 1 \quad \forall n \geq 3$

$\Downarrow$

$\frac{n}{1 - 7\frac{1}{n^2}} > n$

$\forall n \geq 3$

$\Rightarrow \dfrac{1}{\frac{1}{n} - 7\frac{1}{n^2}}$ is unbounded

So $(a_n)_{n=1}^{\infty}$ is unbounded

$\Rightarrow$ diverges $\#$

<u>Thm</u> (Pinching thm).
Suppose for all $n$ sufficiently large,

$$a_n \le b_n \le c_n.$$

If $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n$,

then

$$\lim_{n\to\infty} b_n = L.$$

<u>Example</u>

① $\lim \dfrac{\cos n}{n} = 0$ because

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

$$-\frac{1}{n} \leq \frac{\cos n}{n} \leq \frac{1}{n}$$

$\forall n \geq 1$

and $\lim_{n\to\infty}\left(-\frac{1}{n}\right) = 0 = \lim_{n\to\infty} \frac{1}{n}$ #

② $\lim_{n\to\infty} \sqrt{4+\left(\frac{1}{n}\right)^2} = 2$ because

$\forall n \geq 1$

$$2 = \sqrt{4} \leq \sqrt{4+\left(\frac{1}{n}\right)^2} \leq \sqrt{\underbrace{4+2\cdot\frac{1}{n}+\left(\frac{1}{n}\right)^2}_{\left(2+\frac{1}{n}\right)^2}}$$

$$= 2+\frac{1}{n} \to 2 \text{ as } n\to\infty$$

#

**Thm** ( Thm 11.3.12)

Suppose $\lim_{n\to\infty} C_n = C$. If a function

$f$ is continuous at $c$, then

$$\lim_{n\to\infty} f(c_n) = f(c)$$
$$= f\left(\lim_{n\to\infty} c_n\right)$$

Example

- $\lim_{n\to\infty} \sin\left(\frac{\pi}{n}\right) = \sin\left(\lim_{n\to\infty} \frac{\pi}{n}\right)$

  $= \sin 0 = 0$ #

- $\lim_{n\to\infty} e^{\frac{\pi}{n}} = e^{\lim_{n\to\infty} \frac{\pi}{n}} = e^{0} = 1$ #

- $\lim_{n\to\infty} \tan^{-1}\left(\frac{\pi}{n}\right) = \tan^{-1}\left(\lim_{n\to\infty} \frac{\pi}{n}\right)$

  $= \tan^{-1} 0 = 0$ #

- $\lim_{n\to\infty} \sqrt{\underbrace{4+\frac{1}{n^2}}_{\to 4}} = \sqrt{4} = 2$ #

- $$\lim_{n\to\infty} \ln\left(\frac{2n-1}{n}\right) = \ln 2$$ #

$\frac{2}{1} = 2 > 0$

- $$\lim_{n\to\infty} \left|\frac{2n-1}{n}\right| = |2| = 2$$ #