Calculus 2/27

Recall

We consider 3 types of limits in this course:

- function type A:

$$\lim_{x \to c} f(x)$$

- function type B:

$$\lim_{x \to \infty} f(x) \quad , \quad \lim_{x \to -\infty} f(x)$$

- sequence type:

$$\lim_{n \to \infty} a_n$$ ← start here

數列

Limit of sequence

Def

A sequence of real numbers is

real numbers

$$a_1, a_2, a_3, \dots = (a_n)_{n=1}^{\infty}$$

Let

$$(a_n)_{n=1}^{\infty} \quad \text{and} \quad (b_n)_{n=1}^{\infty}$$

be sequences, $\alpha \in \mathbb{R}$.

- scalar product:

$$(\alpha a_n)_{n=1}^{\infty} = \alpha \cdot a_1, \ \alpha \cdot a_2, \ \alpha \cdot a_3, \dots$$

- sum:

$$(a_n + b_n)_{n=1}^{\infty} = a_1 + b_1, \ a_2 + b_2, \ a_3 + b_3, \dots$$

- difference:

$$(a_n - b_n)_{n=1}^{\infty} = a_1 - b_1, \ a_2 - b_2, \dots$$

- product:

$$(a_n \cdot b_n)_{n=1}^{\infty} = a_1 \cdot b_1, \ a_2 \cdot b_2, \dots$$

Assume $b_n \neq 0$ for all n

- quotient: Assume $b_n \neq 0$ for all n

$$\left(\frac{a_n}{b_n}\right)_{n=1}^{\infty} = \frac{a_1}{b_1}, \frac{a_2}{b_2}, \cdots$$

<u>Example</u>

NOT bounded above 等差

① $(a_n = n)_{n=1}^{\infty} = 1, \overset{1}{\frown} 2, \overset{1}{\frown} 3, 4, \cdots$

bounded above by $M=1$ $\left(\frac{1}{n} \leq 1 \quad \forall n = 1, 2, \cdots\right)$

② $\left(\frac{1}{a_n}\right)_{n=1}^{\infty} = \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \cdots$

bounded below by $N=0$ $\left(\frac{1}{n} \geq 0 \quad \forall n = 1, 2, \cdots\right)$

③ $(2^n)_{n=1}^{\infty} = 2, 4, 8, 16, \cdots$

④ $1, 1, 2, 3, 5, 8, 13, \cdots = (b_n)_{n=1}^{\infty}$

$$b_{n+2} = b_{n+1} + b_n$$

<u>Def</u>

We say that a sequence $(a_n)_{n=1}^{\infty}$ is

- 有上界 <u>bounded above</u> if $\exists M$ s.t.

$$a_n \leq M \quad \forall n,$$

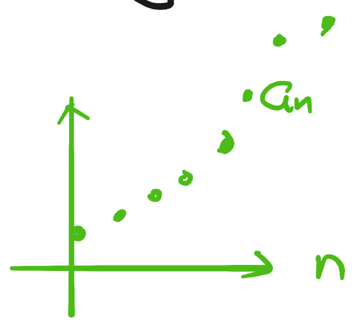
- 有下界 bounded below if $\exists N$ s.t.

$$a_n \geq N \quad \forall n,$$

有界

- bounded if it is both bounded above and below

- strictly increasing (= increasing in book)
  if
  $$a_n < a_{n+1} \quad \forall n$$



function version:

$$f(x) < f(y) \quad \forall\, x < y$$

- increasing (= nondecreasing in book)
  if
  $$a_n \leq a_{n+1} \quad \forall n$$

- strictly decreasing (= decreasing in book)
  if
  $$a_n > a_{n+1} \quad \forall n$$

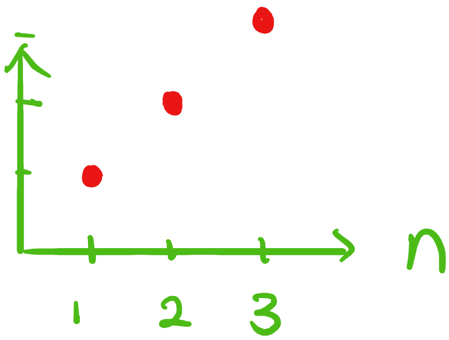
- decreasing (= nonincreasing in book)

if

$$a_n \geq a_{n+1} \quad \forall n$$

<u>Example</u>

① $(a_n = n)_{n=1}^{\infty}$ is strictly increasing and bounded below



② $(b_n = 1)_{n=1}^{\infty}$ is bounded.

$1, 1, 1, 1, \cdots$

③ $(a_n + b_n)_{n=1}^{\infty} = (n+1)_{n=1}^{\infty}$

$= 2, 3, 4, 5, \cdots$

④ $\left(\frac{a_n}{a_n + b_n}\right)_{n=1}^{\infty} = \left(\frac{n}{n+1}\right)_{n=1}^{\infty}$

$= \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \cdots$

$= \frac{}{2}, \frac{}{3}, \frac{}{4}$

is strictly increasing and bounded

Recall:

$e = 2, \dots$

$(e^x)' = e^x$

$\int_1^e \frac{1}{t}\,dt = 1$

⑤ $\left(C_n = \frac{n}{e^n}\right)_{n=1}^{\infty}$

$= \frac{1}{e}, \frac{2}{e^2}, \frac{3}{e^3}, \dots$

Q: increasing? decreasing?

<u>Sol</u>

Let

$$f(x) = \frac{x}{e^x} = x \cdot e^{-x}$$

$$\Rightarrow \quad C_n = f(n) \qquad n = 1, 2, 3, \dots$$

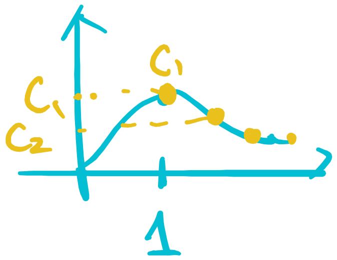
Since

$$f'(x) = (x \cdot e^{-x})'$$

$$= (x)' \cdot e^{-x} + x \cdot (e^{-x})'$$

$f(x) = \frac{x}{e^x}$

$= -1$



$$= 1 \cdot e^{-x} + x e^{-x} \cdot (-x)'$$

$$= e^{-x} - x e^{-x}$$

$$= \underbrace{(1-x)}_{\leq 0 \text{ if } x \geq 1} e^{-x} \leq 0 \quad \forall x \geq 1$$

>0

we have

$$\underset{c_{n+1}}{\underset{\parallel}{f(n+1)}} \leq \underset{c_n}{\underset{\parallel}{f(n)}} \quad \forall n = 1, 2, \ldots$$

So

$$\left(c_n = \frac{n}{e^n} \right)_{n=1}^{\infty}$$

is decreasing #

<u>Def</u>

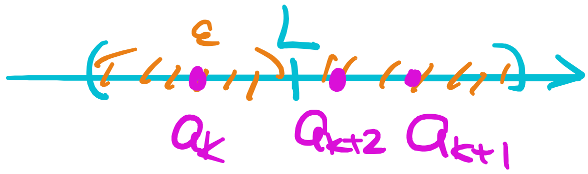
A sequence $(a_n)_{n=1}^{\infty}$ is Convergent if there exists a number L with the property:

"$\forall\, \varepsilon > 0 \quad \exists\, K \quad$ s.t. $\qquad a_n \in (L-\varepsilon,\, L+\varepsilon)$

$\Updownarrow$

if $n \geq K$, then $\quad |a_n - L| < \varepsilon$"

$a_K,\ a_{K+1},\ a_{K+2},\ \ldots$



In this case, we say L is the limit of $(a_n)_{n=1}^{\infty}$, denoted by

$$\lim_{n\to\infty} a_n = L$$

or

$$a_n \longrightarrow L \qquad (\text{as } n \to \infty)$$

We say $(a_n)_{n=1}^{\infty}$ is divergent if it is NOT convergent.

Example

Prove that

$$\lim_{n \to \infty} \frac{1}{n} = 0$$

<u>pf</u>

Given any $\varepsilon > 0$, there exists

$$K = \frac{1}{\varepsilon} + 1$$

s.t. if $n \geq K$, then

$$\left| \frac{1}{n} - 0 \right| = \frac{1}{n} \leq \frac{1}{K}$$

$$= \frac{1}{\frac{1}{\varepsilon} + 1} < \frac{1}{\frac{1}{\varepsilon}} = \varepsilon \qquad \#$$

<u>Thm</u> (§11.3)

① The limit is unique.

② Every convergent sequence is bounded.

③ Every unbounded sequence is divergent.

e.g. $(a_n = n)_{n=1}^{\infty} = 1, 2, 3, \dots$

is divergent.