Calculus — Homework 8 (Spring 2024)

1. Find the gradient.

   (a) $f(x, y) = 3x^2 - xy + y$.

   (b) $f(x, y) = x^2 e^{-y}$.

   (c) $f(x, y, z) = z \sin(x - y)$.

   (d) $f(x, y, z) = z^{xy^2}$.

2. Find the gradient at $\vec{p}$.

   (a) $f(x, y) = 2x^2 - 3xy + 4y^2; \quad \vec{p} = (2, 3)$.

   (b) $f(x, y) = 2x(x - y)^{-1}; \quad \vec{p} = (3, 1)$.

   (c) $f(x, y, z) = e^{-x} \sin(z + 2y); \quad \vec{p} = (0, \frac{1}{4}\pi, \frac{1}{4}\pi)$.

   (d) $f(x, y, z) = \cos(xyz^2); \quad \vec{p} = (\pi, \frac{1}{4}, -1)$.

3. Find the directional derivative at the point $\vec{p}$ in the direction $\vec{u}$.

   (a) $f(x, y) = x^2 + 3y^2; \quad \vec{p} = (1, 1), \quad \vec{u} = \frac{1}{\sqrt{2}}(1, -1)$.

   (b) $f(x, y) = x^2 y + \tan y; \quad \vec{p} = (-1, \pi/4), \quad \vec{u} = \frac{1}{\sqrt{5}}(1, -2)$.

   (c) $f(x, y, z) = xy + yz + zx; \quad \vec{p} = (1, -1, 1), \quad \vec{u} = \frac{1}{\sqrt{6}}(1, 2, 1)$.

   (d) $f(x, y, z) = (x + y^2 + z^3)^2; \quad \vec{p} = (1, -1, 1), \quad \vec{u} = \frac{1}{\sqrt{2}}(1, 1, 0)$.

4. Let $f(x, y) = x^3 - xy$. Set $\vec{a} = (0, 1)$ and $\vec{b} = (1, 3)$. Find a point $\vec{c}$ on the line segment connecting $\vec{a}$ and $\vec{b}$ for which
$$f(\vec{b}) - f(\vec{a}) = \nabla f(\vec{c}) \cdot (\vec{b} - \vec{a}).$$

5. Let f be a smooth function on $\mathbb{R}^3$. Show that if $f(\vec{a}) = f(\vec{b})$, then there exists a point $\vec{c}$ between $\vec{a}$ and $\vec{b}$ for which $\nabla f(\vec{c}) \perp (\vec{b} - \vec{a})$.

6. Find the rate of change of f with respect to t along the curve $\vec{\gamma}$.

   (a) $f(x, y) = x^2 y, \quad \vec{\gamma}(t) = e^t \,\vec{\imath} + e^{-t} \,\vec{\jmath}$.

   (b) $f(x, y) = \arctan(y^2 - x^2), \quad \vec{\gamma}(t) = \sin t \,\vec{\imath} + \cos t \,\vec{\jmath}$.

   (c) $f(x, y, z) = \ln(x^2 + y^2 + z^2), \quad \vec{\gamma}(t) = \sin t \,\vec{\imath} + \cos t \,\vec{\jmath} + e^{2t} \,\vec{k}$.

   (d) $f(x, y, z) = y \sin(x + z), \quad \vec{\gamma}(t) = 2t \,\vec{\imath} + \cos t \,\vec{\jmath} + t^3 \,\vec{k}$.

7. Find $\partial u/\partial s$ and $\partial u/\partial t$.

   (a) $u = x^2 - xy; \quad x = s \cos t,\ y = t \sin s$.

   (b) $u = x^2 \tan y; \quad x = s^2 t,\ y = s + t^2$.

   (c) $u = z^2 \sec(xy); \quad x = 2st,\ y = s - t^2,\ z = s^2 t$.

   (d) $u = xe^{yz^2}; \quad x = \ln(st),\ y = t^3,\ z = s^2 + t^2$.

8. Let f be a continuously differentiable function of one variable, and
$$r = \|\,\vec{x}\,\| = \sqrt{x^2 + y^2 + z^2}.$$
Show that
$$\nabla\big(f(r)\big) = \nabla\big(f(\|\,\vec{x}\,\|)\big) = f'(r)\frac{\vec{x}}{r}, \qquad \forall \vec{x} \neq \vec{0}.$$

9. Let
$$x = r\cos\theta \qquad \text{and} \qquad y = r\sin\theta.$$
Suppose $u = u(x, y)$ is a smooth function.

   (a) Show that
$$\nabla u = \frac{\partial u}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial u}{\partial \theta}\vec{e}_\theta,$$
   where $r \neq 0$,
$$\vec{e}_r = \cos\theta\,\vec{\imath} + \sin\theta\,\vec{\jmath} \qquad \text{and} \qquad \vec{e}_\theta = -\sin\theta\,\vec{\imath} + \cos\theta\,\vec{\jmath}.$$

   (b) Show that
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r}\frac{\partial u}{\partial r}.$$