Calculus — Homework 7 (Spring 2024)

1. Find the length of the curve.
   (a) $\vec{\gamma}(t) = t\,\vec{\imath} + \frac{2}{3}t^{3/2}\,\vec{\jmath}$, from $t = 0$ to $t = 8$.
   (b) $\vec{\gamma}(t) = e^t\big(\cos t\,\vec{\imath} + \sin t\,\vec{\jmath}\big)$, from $t = 0$ to $t = \pi$.
   (c) $\vec{\gamma}(t) = t\,\vec{\imath} + \ln(\sec t)\,\vec{\jmath} + 3\,\vec{k}$, from $t = 0$ to $t = \frac{\pi}{4}$.
   (d) $\vec{\gamma}(t) = (t\sin t + \cos t)\,\vec{\imath} + (\sin t - t\cos t)\,\vec{\jmath} + \frac{1}{2}\sqrt{3}\,t^2\,\vec{k}$, from $t = 0$ to $t = 2\pi$.

2. Let
$$\vec{\gamma}(t) = 3\cos t\,\vec{\imath} + 3\sin t\,\vec{\jmath} + 4t\,\vec{k}, \qquad t \geq 0.$$
   (a) Let
$$s(t) = \int_0^t \|\,\vec{\gamma}'(u)\,\|\,du.$$
   Show that s is a one-to-one function, and find its inverse function $\tau(s)$.
   (b) Let
$$\vec{R}(s) = \vec{\gamma}(\tau(s)).$$
   Show that
$$\Big\|\,\frac{d\vec{R}}{ds}\,\Big\| = 1.$$
   (The parametrization s is called the **parametrization by arc length**.)
   (c) Find the length of the curve
$$\vec{R}(s), \qquad 0 \leq s \leq L.$$

3. Calculate the first order and second order partial derivatives.
   (a) $f(x,y) = 3x^2 - xy + y$.
   (b) $f(x,y) = x^2e^{-y}$.
   (c) $f(x,y,z) = z\sin(x-y)$.
   (d) $f(x,y,z) = z^{xy^2}$.

4. Calculate.
   (a) Find $f_x(0,e)$, $f_y(0,e)$, $f_{xy}(0,e)$, $f_{xxx}(0,e)$ and $f_{xyx}(0,e)$ given that $f(x,y) = e^x\ln y$.
   (b) Find $f_x(0,\frac{1}{4}\pi)$, $f_y(0,\frac{1}{4}\pi)$, $f_{yy}(0,\frac{1}{4}\pi)$ and $f_{xyxy}(0,\frac{1}{4}\pi)$ given that $f(x,y) = e^{-x}\sin(x+2y)$.
   (c) Find $f_x(1,2)$, $f_y(1,2)$ and $f_{xx}(1,2)$ given that $f(x,y) = \dfrac{x}{x+y^2}$.

5. Show that the functions u and v satisfy the **Cauchy–Riemann equations**
$$u_x(x,y) = v_y(x,y) \qquad \text{and} \qquad u_y(x,y) = -v_x(x,y).$$
   These equations are fundamentally important in the study of functions of a complex variable.
   (a) $u(x,y) = x^2 - y^2$; $v(x,y) = 2xy$. (Also compute $(x + y\sqrt{-1})^2$.)
   (b) $u(x,y) = e^x\cos y$; $v(x,y) = e^x\sin y$.
   (c) $u(x,y) = \frac{1}{2}\ln(x^2+y^2)$; $v(x,y) = \arctan\dfrac{y}{x}$.
   (d) $u(x,y) = \dfrac{x}{x^2+y^2}$; $v(x,y) = \dfrac{-y}{x^2+y^2}$. (Also compute $\dfrac{1}{(x+y\sqrt{-1})}$.)

6. Let g be a twice differentiable function of one variable and set
$$f(x,y) = g(x+y) + g(x-y).$$
   Show that
$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial y^2}.$$

7. Let f be a smooth function of two variables. Show that

$$\frac{\partial^3 f}{\partial x^2 \partial y} = \frac{\partial^3 f}{\partial x \partial y \partial x} = \frac{\partial^3 f}{\partial y \partial x^2}.$$

8. Set

$$f(x,y) = \begin{cases} \dfrac{xy(y^2 - x^2)}{x^2 + y^2}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$$

   (a) Find the second order partial derivatives of f.

   (b) Show that

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) \neq \frac{\partial^2 f}{\partial y \partial x}(0,0).$$