Calculus — Homework 6 (Spring 2024)

1. Simplify.

   (a) $(3\vec{a} \cdot \vec{b}) - (\vec{a} \cdot 2\vec{b})$.

   (b) $\vec{a} \cdot (\vec{a} - \vec{b}) + \vec{b} \cdot (\vec{b} + \vec{a})$.

   (c) $(\vec{a} - \vec{b}) \cdot \vec{c} + \vec{b} \cdot (\vec{c} + \vec{a})$.

   (d) $\vec{a} \cdot (\vec{a} + 2\vec{c}) + (2\vec{b} - \vec{a}) \cdot (\vec{a} + 2\vec{c}) - 2\vec{b} \cdot (\vec{a} + 2\vec{c})$.

2. Consider vectors in $\mathbb{R}^3$.

   (a) Find $\vec{a}, \vec{b}, \vec{c} \neq \vec{0}$ such that $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c}$ but $\vec{b} \neq \vec{c}$.

   (b) Show that if $\vec{u} \cdot \vec{b} = \vec{u} \cdot \vec{c}$ for all *unit vectors* $\vec{u}$ (i.e. $\|\vec{u}\| = 1$), then $\vec{b} = \vec{c}$.

3. Let $\vec{a}, \vec{b} \in \mathbb{R}^3$.

   (a) Show that for all vectors $\vec{a}$ and $\vec{b}$

   $$4(\vec{a} \cdot \vec{b}) = \|\vec{a} + \vec{b}\|^2 - \|\vec{a} - \vec{b}\|^2.$$

   (b) Show that $\vec{a} \perp \vec{b}$ iff $\|\vec{a} + \vec{b}\| = \|\vec{a} - \vec{b}\|$.

   (c) Show that, if $\vec{a}$ and $\vec{b}$ are nonzero vectors such that

   $$(\vec{a} + \vec{b}) \perp (\vec{a} - \vec{b}) \qquad \text{and} \qquad \|\vec{a} + \vec{b}\| = \|\vec{a} - \vec{b}\|,$$

   then the parallelogram generated by $\vec{a}$ and $\vec{b}$ is a square.

4. Show that

   $$|\vec{a} \cdot \vec{b}| \leq \|\vec{a}\|\|\vec{b}\|,$$

   and the equality holds iff $\vec{a}$ and $\vec{b}$ are *parallel*, i.e. there exists λ such that $\vec{a} = \lambda\vec{b}$ or $\vec{b} = \lambda\vec{a}$.

5. Prove the *parallelogram law*:

   $$\|\vec{a} + \vec{b}\|^2 + \|\vec{a} - \vec{b}\|^2 = 2\|\vec{a}\|^2 + 2\|\vec{b}\|^2.$$

6. Calculate the $\vec{f}'(t)$ and $\vec{f}''(t)$.

   (a) $\vec{f}(t) = (1 + 2t)\,\vec{\imath} + (3 - t)\,\vec{\jmath} + \cos t\vec{k}$.

   (b) $\vec{f}(t) = e^t(\vec{\imath} - \vec{\jmath}) + e^{-2t}(\vec{\jmath} - \vec{k})$.

   (c) $\vec{f}(t) = \big((t^2\,\vec{\imath} - \vec{\jmath}) \cdot (\vec{\imath} - t^2\,\vec{\jmath})\big)\,\vec{\imath}$.

7. Find $\lim\limits_{t \to 0} \vec{f}(t)$ if it exists. Explain why if the limit does not exist.

   (a) $\vec{f}(t) = (1 + 2t)\,\vec{\imath} + (3 - t)\,\vec{\jmath} + \dfrac{t}{|t|}\vec{k}$.

   (b) $\vec{f}(t) = e^t(\vec{\imath} - \vec{\jmath}) + e^{-2t}(\vec{\jmath} - \vec{k})$.

   (c) $\vec{f}(t) = \dfrac{\sin t}{2t}\,\vec{\imath} + e^{2t}\,\vec{\jmath} + \dfrac{t^2}{e^t}\,\vec{k}$.

   (d) $\vec{f}(t) = t^2\,\vec{\imath} + \dfrac{1 - \cos t}{3t}\,\vec{\jmath} + \dfrac{t}{t + 1}\,\vec{k}$.

8. Let $\vec{f}$ be a differentiable vector-valued function. Show that if $\|\vec{f}(t)\| \neq 0$, then

   $$\frac{d}{dt}(\|\vec{f}(t)\|) = \frac{\vec{f}(t) \cdot \vec{f}'(t)}{\|\vec{f}(t)\|}.$$

9. Suppose $\vec{\gamma}(t)$ is a differentiable vector-valued function. Show that $\|\vec{\gamma}(t)\|$ is constant iff $\vec{\gamma}(t) \cdot \vec{\gamma}'(t) = 0$ for all t.