

## Calculus — Homework 6 (Spring 2024)

1. Simplify.

(a)  $(3\vec{a} \cdot \vec{b}) - (\vec{a} \cdot 2\vec{b})$ .

(c)  $(\vec{a} - \vec{b}) \cdot \vec{c} + \vec{b} \cdot (\vec{c} + \vec{a})$ .

(b)  $\vec{a} \cdot (\vec{a} - \vec{b}) + \vec{b} \cdot (\vec{b} + \vec{a})$ .

(d)  $\vec{a} \cdot (\vec{a} + 2\vec{c}) + (2\vec{b} - \vec{a}) \cdot (\vec{a} + 2\vec{c}) - 2\vec{b} \cdot (\vec{a} + 2\vec{c})$ .

2. Consider vectors in  $\mathbb{R}^3$ .

(a) Find  $\vec{a}, \vec{b}, \vec{c} \neq \vec{0}$  such that  $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c}$  but  $\vec{b} \neq \vec{c}$ .

(b) Show that if  $\vec{u} \cdot \vec{b} = \vec{u} \cdot \vec{c}$  for all *unit vectors*  $\vec{u}$  (i.e.  $\|\vec{u}\| = 1$ ), then  $\vec{b} = \vec{c}$ .

3. Let  $\vec{a}, \vec{b} \in \mathbb{R}^3$ .

(a) Show that for all vectors  $\vec{a}$  and  $\vec{b}$

$$4(\vec{a} \cdot \vec{b}) = \|\vec{a} + \vec{b}\|^2 - \|\vec{a} - \vec{b}\|^2.$$

(b) Show that  $\vec{a} \perp \vec{b}$  iff  $\|\vec{a} + \vec{b}\| = \|\vec{a} - \vec{b}\|$ .

(c) Show that, if  $\vec{a}$  and  $\vec{b}$  are nonzero vectors such that

$$(\vec{a} + \vec{b}) \perp (\vec{a} - \vec{b}) \quad \text{and} \quad \|\vec{a} + \vec{b}\| = \|\vec{a} - \vec{b}\|,$$

then the parallelogram generated by  $\vec{a}$  and  $\vec{b}$  is a square.

4. Show that

$$|\vec{a} \cdot \vec{b}| \leq \|\vec{a}\| \|\vec{b}\|,$$

and the equality holds iff  $\vec{a}$  and  $\vec{b}$  are *parallel*, i.e. there exists  $\lambda$  such that  $\vec{a} = \lambda\vec{b}$  or  $\vec{b} = \lambda\vec{a}$ .

5. Prove the *parallelogram law*:

$$\|\vec{a} + \vec{b}\|^2 + \|\vec{a} - \vec{b}\|^2 = 2\|\vec{a}\|^2 + 2\|\vec{b}\|^2.$$

6. Calculate the  $\vec{f}'(t)$  and  $\vec{f}''(t)$ .

(a)  $\vec{f}(t) = (1 + 2t)\vec{i} + (3 - t)\vec{j} + \cos t\vec{k}$ .

(b)  $\vec{f}(t) = e^t(\vec{i} - \vec{j}) + e^{-2t}(\vec{j} - \vec{k})$ .

(c)  $\vec{f}(t) = ((t^2\vec{i} - \vec{j}) \cdot (\vec{i} - t^2\vec{j}))\vec{i}$ .

7. Find  $\lim_{t \rightarrow 0} \vec{f}'(t)$  if it exists. Explain why if the limit does not exist.

(a)  $\vec{f}(t) = (1 + 2t)\vec{i} + (3 - t)\vec{j} + \frac{t}{|t|}\vec{k}$ .

(b)  $\vec{f}(t) = e^t(\vec{i} - \vec{j}) + e^{-2t}(\vec{j} - \vec{k})$ .

(c)  $\vec{f}(t) = \frac{\sin t}{2t}\vec{i} + e^{2t}\vec{j} + \frac{t^2}{e^t}\vec{k}$ .

(d)  $\vec{f}(t) = t^2\vec{i} + \frac{1 - \cos t}{3t}\vec{j} + \frac{t}{t+1}\vec{k}$ .

8. Let  $\vec{f}$  be a differentiable vector-valued function. Show that if  $\|\vec{f}'(t)\| \neq 0$ , then

$$\frac{d}{dt}(\|\vec{f}(t)\|) = \frac{\vec{f}(t) \cdot \vec{f}'(t)}{\|\vec{f}(t)\|}.$$

9. Show that  $\|\vec{\gamma}(t)\|$  is constant iff  $\vec{\gamma}(t) \cdot \vec{\gamma}'(t) = 0$  for all  $t$ .