# Calculus — Homework 5 (Spring 2024)

1. Expand $f(x)$ in powers of $x$.

   (a) $f(x) = \dfrac{1}{(1-x)^2}$.

   (b) $f(x) = \ln(2-3x)$.

   (c) $f(x) = \sin(x^2)$.

   (d) $f(x) = x^2 \arctan x$.

   (e) $f(x) = \dfrac{1}{1-x} + e^{2x^3}$.

   (f) $f(x) = \cosh x \sinh x$.

2. Find $f^{(9)}(0)$.

   (a) $f(x) = \ln(2-3x)$.

   (b) $f(x) = x\sin(x^2)$.

   (c) $f(x) = x^2 \arctan x$.

   (d) $f(x) = \dfrac{1}{1-x} + e^{2x^3}$.

3. Evaluate the limit by using power series.

   (a) $\displaystyle\lim_{x\to 0} \frac{1-\cos x}{x^2}$.

   (b) $\displaystyle\lim_{x\to 0} \frac{\sin x - x}{x^2}$.

4. Sum the series.

   (a) $\displaystyle\sum_{k=0}^{\infty} \frac{1}{k!} x^{3k+1}$.

   (b) $\displaystyle\sum_{k=0}^{\infty} \frac{3k}{k!} x^{3k-1}$.

5. Set $f(x) = \dfrac{e^x - 1}{x}$.

   (a) Expand $f(x)$ in a power series.

   (b) Differentiate the series and show that

$$\sum_{n=1}^{\infty} \frac{n}{(n+1)!} = 1.$$

6. Show that, if $\displaystyle\sum_{k=0}^{\infty} a_k x^k$ and $\displaystyle\sum_{k=0}^{\infty} b_k x^k$ both converge to the same sum on $(-r, r)$, $r > 0$, then $a_k = b_k$ for each $k$.

7. Let $\alpha$ be an arbitrary real number (not necessarily an integer). Set

$$f(x) = (1+x)^\alpha.$$

   (a) Show that the Taylor series of $f(x)$ can be written as

$$1 + \sum_{k=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} x^k = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!} x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!} x^3 + \cdots$$

   which is called the **binomial series**.

   (b) Show that the binomial series converges absolutely on $(-1, 1)$.

   (c) Let

$$\varphi(x) = 1 + \sum_{k=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} x^k, \qquad \forall x \in (-1,1).$$

   Use term-by-term differentiation to show that

$$(1+x)\varphi'(x) = \alpha\varphi(x), \qquad \forall x \in (-1,1),$$
$$\varphi(0) = 1.$$

(d) Prove that

$$f(x) = \varphi(x), \qquad \forall x \in (-1, 1).$$

(Hint: Differentiate $\dfrac{\varphi(x)}{(1+x)^\alpha}$.)

That is,

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \cdots$$

for all $x \in (-1, 1)$.

In particular, we have

$$(1+x)^2 = 1 + 2x + x^2,$$
$$(1+x)^3 = 1 + 3x + 3x^2 + x^3,$$

and

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

if $n$ is a positive integer.