# Calculus — Homework 4 (Spring 2024)

1. Does the series absolutely converge, conditionally converge or diverge? Explain why.

   (a) $\sum_{k=1}^{\infty}(-1)^k$.

   (b) $\sum_{k=2}^{\infty}\frac{(-1)^k}{k\ln k}$.

   (c) $\sum_{k=2}^{\infty}(-1)^k\frac{\ln k}{k}$.

   (d) $\sum_{k=2}^{\infty}(-1)^k\frac{k}{\ln k}$.

   (e) $\sum_{k=1}^{\infty}(-1)^k k\sin(1/k)$.

   (f) $\sum_{k=1}^{\infty}\frac{\sin(\pi k/2)}{k\sqrt{k}}$.

2. Let $f$ be a function which can be differentiated infinitely many times on $(-1,1)$. The **$n$-th Taylor polynomial** of $f(x)$ is

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n.$$

   The **$n$-th remainder** of $f(x)$ is

$$R_n(x) = f(x) - P_n(x).$$

   (a) Prove that

$$R_2(x) = \frac{1}{2}\int_0^x f^{(3)}(t)\cdot(x-t)^2 dt$$

   for each $x \in (-1,1)$.

   (b) Find $P_4$ for $f(x) = \sqrt{1+x}$.

   (c) Show that if $f(x) = \sqrt{1+x}$, then

$$|R_2(x)| < \frac{\sqrt{2}}{32}, \qquad \forall x \in (-1/2, 1/2).$$

3. Expand $g(x)$ in the powers of $x-1$ and specify the values of $x$ for which the expansion is valid.

   (a) $g(x) = 3x^3 - 2x^2 + 4x + 1$.

   (b) $g(x) = x^{-1}$.

   (c) $g(x) = e^{-4x}$.

   (d) $g(x) = \sin \pi x$.

   (e) $g(x) = \cos(\frac{1}{2}\pi x)$.

   (f) $g(x) = \ln(1+2x)$.

4. Suppose that the series $\sum_{k=0}^{\infty} a_k(-3)^k$ converges. What can you conclude about the convergence of the following series?

   (a) $\sum_{k=1}^{\infty} a_k 2^k$.

   (b) $\sum_{k=2}^{\infty} a_k 3^k$.

   (c) $\sum_{k=3}^{\infty}(-1)^k|a_k|$.

   (d) $\sum_{k=4}^{\infty} a_k 4^k$.

5. Find the interval of convergence.

   (a) $\sum_{k=1}^{\infty} kx^k$.

   (b) $\sum_{k=1}^{\infty}\frac{1}{(2k)!}x^{4k}$.

   (c) $\sum_{k=1}^{\infty}(-k)^{2k}x^{2k}$.

   (d) $\sum_{k=1}^{\infty}\frac{1}{k2^k}x^k$.

   (e) $\sum_{k=1}^{\infty}\frac{\ln k}{k}(x+1)^k$.

   (f) $\sum_{k=1}^{\infty}(-1)^k\frac{k!}{k^3}(x-1)^k$.

   (g) $\sum_{k=1}^{\infty}\frac{\ln k}{2^k}(x-2)^k$.

   (h) $\sum_{k=2}^{\infty}\frac{1}{(\ln k)^k}(x-1)^k$.

   (i) $\sum_{k=1}^{\infty}\left(1+\frac{1}{k}\right)^k(x+2)^k$.