Calculus — Homework 3 (Spring 2024)

1. Evaluate the integrals.

   (a) $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}}$.

   (b) $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^2}}$.

   (c) $\displaystyle\int_0^1 x \ln x \, dx$.

2. Does the integral converge or diverge?

   (a) $\displaystyle\int_{1/2}^2 \frac{dx}{x \ln x}$.

   (b) $\displaystyle\int_0^{\pi/2} \tan x \, dx$.

   (c) $\displaystyle\int_1^\infty \frac{dx}{x^3+1}$.

   (d) $\displaystyle\int_\infty^1 \frac{\sqrt{x+1}}{x^2} \, dx$.

   (e) $\displaystyle\int_\pi^\infty \frac{2+\cos x}{x} \, dx$.

   (f) $\displaystyle\int_1^\infty \frac{e^x}{x} \, dx$.

3. Determine whether the series converges or diverges. If it converges, find the sum of the series. If it diverges, explain why.

   (a) $\displaystyle\sum_{k=3}^\infty \frac{1}{k^2-k}$.

   (b) $\displaystyle\sum_{k=0}^\infty \frac{(-1)^k}{5^k}$.

   (c) $\displaystyle\sum_{k=0}^\infty \frac{3^{k-1}}{4^{3k+1}}$.

   (d) $\displaystyle\sum_{k=1}^\infty \left(\frac{k-1}{k}\right)^k$.

4. Let j be a positive integer.

   (a) Show that
   $$\sum_{k=0}^\infty a_k \quad \text{converges} \qquad \text{iff} \qquad \sum_{k=j}^\infty a_k \quad \text{converges.}$$

   (b) Show that if $\displaystyle\sum_{k=0}^\infty a_k = L$, then $\displaystyle\sum_{k=j}^\infty a_k = L - \sum_{k=0}^{j-1} a_k$.

   (c) Show that if $\displaystyle\sum_{k=j}^\infty a_k = M$, then $\displaystyle\sum_{k=0}^\infty a_k = M + \sum_{k=0}^{j-1} a_k$.

5. Determine whether the series converges or diverges. Explain why.

   (a) $\displaystyle\sum_{k=2}^\infty \frac{k}{k^3-k}$.

   (b) $\displaystyle\sum_{k=1}^\infty \frac{\ln k}{k}$.

   (c) $\displaystyle\sum_{k=1}^\infty \frac{\ln k}{k^2}$.

   (d) $\displaystyle\sum_{k=1}^\infty \frac{1}{1+2\ln k}$.

   (e) $\displaystyle\sum_{k=1}^\infty k^2 2^{-k^3}$.

   (f) $\displaystyle\sum_{k=1}^\infty \frac{2+\cos k}{\sqrt{k+1}}$.

   (g) $\displaystyle\sum_{k=1}^\infty \frac{10^k}{k!}$.

   (h) $\displaystyle\sum_{k=1}^\infty \frac{1}{k 2^k}$.

   (i) $\displaystyle\sum_{k=1}^\infty \frac{k^2}{e^k}$.

   (j) $\displaystyle\sum_{k=1}^\infty \frac{k^k}{3^{k^2}}$.

   (k) $\displaystyle\sum_{k=1}^\infty \frac{2 \cdot 4 \cdots 2k}{(2k)!}$.

   (l) $\displaystyle\sum_{k=1}^\infty \frac{k!}{k^{k/2}}$.