Calculus — Homework 1 (Spring 2024)

1. Determine the boundedness and monotonicity (i.e. increasing or decreasing or neither) of the sequence with a_n, as indicated.

   (a) $a_n = \frac{2}{n}$.
   (b) $a_n = \frac{(-1)^n}{n}$.
   (c) $a_n = \frac{n + (-1)^n}{n}$.
   (d) $a_n = \sqrt{n^2+1}$.
   (e) $a_n = \frac{2^n}{4^n+1}$.
   (f) $a_n = \frac{1}{2n} - \frac{1}{2n+3}$.
   (g) $a_n = \ln\left(\frac{n+1}{n}\right)$.
   (h) $a_n = \sin\frac{\pi}{n+1}$.
   (i) $a_n = \frac{3^n}{(n+1)^2}$.

2. Let a_n be the sequence satisfying the given rules. Find an explicit formula for a_n that does not involve an recursive relation. Prove your answer.

   (a) $a_1 = 1;\qquad a_{n+1} = \frac{1}{2}a_n + 1$.
   (b) $a_1 = 1;\qquad a_{n+1} = \frac{n}{n+1}a_n$.
   (c) $a_1 = 1;\qquad a_{n+1} = a_n + \frac{1}{n(n+1)}$.
   (d) $a_1 = 1;\qquad a_{n+1} = a_n + \cdots + a_1$.

3. Assume $|r| < 1$. Prove that $\lim_{n\to\infty} r^n = 0$.

4. Let r be a real number, and
$$a_n = 1 + r + r^2 + \cdots + r^{n-1}.$$
   (a) If $r = 1$, what is a_n for $n = 1, 2, 3, \cdots$?
   (b) If $r \neq 1$, what is a_n for $n = 1, 2, 3, \cdots$? Find a formula for a_n that does not involve adding up the powers of r.
   (c) For what values of r does a_n converge?
   (d) Find the limit $\lim_{n\to\infty} a_n$ for $|r| < 1$.

5. State whether the sequence converges and, if it does, find the limit.

   (a) $a_n = 2^n$.
   (b) $a_n = \frac{(-1)^n}{n}$.
   (c) $a_n = \frac{n + (-1)^n}{n}$.
   (d) $a_n = \frac{4^n}{\sqrt{n^2+1}}$.
   (e) $a_n = \frac{2^n}{4^n+1}$.
   (f) $a_n = \frac{1}{2n} - \frac{1}{2n+3}$.
   (g) $a_n = \ln\left(\frac{n+1}{n}\right)$.
   (h) $a_n = \sin\frac{\pi}{n+1}$.
   (i) $a_n = \frac{(n+1)(n+2)}{(n+3)(n+4)}$.