Calculus — Homework 11 (Spring 2024)

1. Set $\Pi : a_1 \le x \le a_2, b_1 \le y \le b_2, c_1 \le z \le c_2$. Show that, if f is continuous on $[a_1, a_2]$, g is continuous on $[b_1, b_2]$, and h is continuous on $[c_1, c_2]$, then

$$\iiint_{\Pi} f(x)g(y)h(z)\,dxdydz = \left(\int_{a_1}^{a_2} f(x)\,dx\right)\cdot\left(\int_{b_1}^{b_2} g(y)\,dy\right)\cdot\left(\int_{c_1}^{c_2} h(z)\,dz\right).$$

2. Evaluate.

   (a) $\displaystyle\int_0^1\int_0^2\int_0^3 dxdydz.$

   (b) $\displaystyle\int_0^1\int_0^x\int_0^y y\,dzdydx.$

   (c) $\displaystyle\int_0^1\int_1^{2y}\int_0^x (x+2z)\,dzdxdy.$

   (d) $\displaystyle\int_1^2\int_y^{y^2}\int_0^{\ln x} ye^z\,dzdxdy.$

3. Evaluate the triple integral.

   (a) $\displaystyle\iiint_T 2ye^x\,dxdydz$, where T is the solid given by $0 \le y \le 1, 0 \le x \le y, 0 \le z \le x+y$.

   (b) $\displaystyle\iiint_T x^2y^2z^2\,dxdydz$, where T is the solid bounded by the planes $z = y+1, y+z = 1, x = 0, x = 1, z = 0$.

4. Find the Jacobian of the transformation.

   (a) $x = u+v, \quad y = 2u - v.$

   (b) $x = uv, \quad y = u^2 + v^2.$

   (c) $x = (1 + w\cos v)\cos u, \quad y = (1 + w\cos v)\sin u, \quad z = w\sin v.$

5. Calculate the area of the region Ω bounded by the curves

$$x^2 - 2xy + y^2 + x + y = 0, \qquad x + y + 4 = 0.$$

   HINT: Set $u = x - y$, $v = x + y$.

6. Calculate the area of the region Ω bounded by the curves

$$x^2 - 4xy + 4y^2 - 2x - y - 1 = 0, \qquad y = \frac{2}{5}.$$

7. Let T be the solid ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1,$$

   where $a, b, c > 0$.

   (a) Calculate the volume of T by setting

$$x = a\rho\sin\phi\cos\theta, \qquad y = b\rho\sin\phi\sin\theta, \qquad z = c\rho\cos\phi.$$

   (b) Evaluate $\displaystyle\iiint_T \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right)dxdydz.$

8. Evaluate the line integral (i) directly by the definition; and (ii) by applying Green's theorem.

   (a) $\displaystyle\oint_C xy\,dx + x^2\,dy$; where C is the triangle with vertices $(0,0)$, $(0,1)$ and $(1,1)$.

   (b) $\displaystyle\oint_C (3x^2 + y)\,dx + (2x + y^3)\,dy$; where C is given by the equation $9x^2 + 4y^2 = 36$.

   (c) $\displaystyle\oint_C y^2\,dx + x^2\,dy$; where C is the boundary of the region that lies between the curves $y = x$ and $y = x^2$.