Calculus 1/4

§ First order linear diff. eq.

$$y' + p(x) \cdot y = g(x) \qquad (*)$$

To solve $(*)$, we consider

$$H(x) = \int p(x)\, dx$$

i.e. choose $H(x)$ s.t.

$$H'(x) = p(x)$$

Multiply $(*)$ by $e^{H(x)}$:

$$= \left(e^{H(x)}\right)'$$

$$e^{H(x)} \cdot y' + \boxed{e^{H(x)} \cdot p(x)}\, y = e^{H(x)} \cdot g(x)$$

$$= \left(e^{H(x)} \cdot y\right)'$$

Thus,

$$e^{H(x)} y = \int e^{H(x)} g(x)\, dx$$

$$\Rightarrow \quad y = e^{-H(x)} \int e^{H(x)} g(x)\, dx$$

$$= e^{-H(x)} \left(\int_a^x e^{H(t)} g(t)\, dt + C \right)$$

#

<u>Example</u>

① Solve

$$y' + \overset{p(x)}{\overset{''}{1}} \cdot y = 1$$

<u>Sol</u>

Consider

$$H(x) = \int p(x)\, dx = x \quad (+c)$$

Multiply the eq. by $e^{H(x)} = e^x$:

$$e^x (y' + y) = \boxed{e^x}$$

$$= e^x \cdot y' + (e^x)' \cdot y \boxed{= (e^x y)'}$$

$$\Rightarrow \quad e^x y = e^x + C$$

$$\Rightarrow \quad y = 1 + C \cdot e^{-x}$$

for some constant C #

② Solve

$$\begin{cases} y' + 2xy = x \\ y(0) = 2 \end{cases}$$

"initial value problem"

sol

Consider

$$H(x) = \int 2x \, dx = x^2 \ (+c)$$

Multiply the eq. by e^{x^2}:

$$e^{x^2} \cdot y' + \boxed{e^{x^2} \cdot 2x} \cdot y = x \cdot e^{x^2}$$

$$\parallel$$

$$(e^{x^2})'$$

$$= (e^{x^2}\cdot y)'$$

$$\Rightarrow \quad e^{x^2}\cdot y = \int \overbrace{x\cdot e^{x^2}}^{\left(\frac{1}{2}e^{x^2}\right)'}\, dx$$

$$= \frac{1}{2}e^{x^2} + C$$

$$\Rightarrow \quad y = \frac{1}{2} + C\cdot e^{-x^2}$$

Solve C by $y(0) = 2$:

$$2 = y(0) = \frac{1}{2} + C\cdot e^{-0^2} = \frac{1}{2} + C$$

$$\Rightarrow \quad C = 2 - \frac{1}{2} = \frac{3}{2}$$

So

$$y(x) = \frac{1}{2} + \frac{3}{2}e^{-x^2} \qquad \#$$

③ (Newton's law of cooling)

Suppose it's $15^\circ C$ and

$T(t)$ = temperature of coffee at time t (minutes)

Assume

$$T(0) = 85^\circ C$$

$$T(3) = 65^\circ C$$

Then how many minutes do you expect to wait for the coffee to cool down to $40^\circ C$?

sol

By Newton,

$$\frac{dT}{dt} = -k(T-15)$$

where k is a constant.

i.e. $T' + kT = 15k$

Consider

$$H(t) \; \left(= \int k\,dt\right) = kt$$

Multiply eq by e^{kt}:

$$e^{kt}\cdot T' + \underbrace{e^{kt}\cdot k}_{=(e^{kt})'}\cdot T = 15k\cdot e^{kt}$$

$$= (e^{kt}\cdot T)'$$

$$\Rightarrow \quad e^{kt}\cdot T = \int 15k e^{kt}\,dt$$

$= 15 e + C$

$\Rightarrow \quad T(t) = 15 + C \cdot e^{-kt}$

(C and k are constant.)

Recall

$$\begin{cases} T(0) = 85 = 15 + C \cdot e^{-k \cdot 0} \\ \qquad\qquad = 15 + C \\ \\ T(3) = 65 = 15 + C \cdot e^{-k \cdot 3} \end{cases}$$

$\Rightarrow \quad C = 85 - 15 = 70$

$\Rightarrow \quad \overset{50}{\cancel{65}} = \cancel{15} + 70 \cdot e^{-3k}$

$\Rightarrow -3k = \ln \frac{5}{7} \Rightarrow k = -\frac{1}{3} \cdot \ln \frac{5}{7}$

≈ 0.11

Want:

$T(t) \approx 15 + 70 e^{-0.11 t} = 40$

$$\Rightarrow \quad t \approx \left(\ln \frac{25}{70}\right) \cdot \frac{1}{-0.11} \approx 9.4$$

(minutes)

#