Calculus 1/2

§ Exponential growth and logistic growth

Malthusian growth model:

Let $P(t)$ be the population at time t and r be the population growth rate

Malthus:

$$\boxed{\frac{dP}{dt} = r \cdot P}$$

Thm (Thm 7.6.1)

If

$(*)$ $\quad f'(t) = k \cdot f(t) \qquad \forall t \in (a,b)$

then there exists $C \in \mathbb{R}$ s.t.

$$\boxed{f(t) = C \cdot e^{kt}} \qquad \forall t \in (a,b).$$

NOTE: $(C \cdot e^{kt})' = C \cdot e^{kt} \cdot k = k(C e^{kt})$

pf

By $(*)$,

$$f'(t) - k \cdot f(t) = 0$$

$$\Rightarrow \quad e^{-kt}\left(f'(t) - k f(t)\right) = 0$$

$$= e^{-kt} f'(t) + \underbrace{(-k e^{-kt})}_{=\,(e^{-kt})'} f(t)$$

$$= \left(e^{-kt} \cdot f(t)\right)' = 0 \quad \forall t \in (a,b)$$

$\Rightarrow \quad \exists\, C \in \mathbb{R}$ s.t.

$$e^{-kt} f(t) = C \qquad \forall t \in (a,b)$$

$$\Rightarrow \quad f(t) = C \cdot e^{kt} \qquad \forall t \in (a,b)$$

#

Thus.

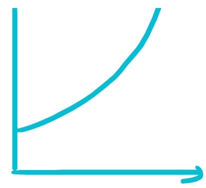
$$\frac{dP}{dt} = rP$$

$$\Rightarrow \quad P(t) = C \cdot e^{rt}$$

$$\Rightarrow \quad P(0) = C \cdot e^{r \cdot 0} = C$$

initial population

$\Rightarrow \quad P(t) = P(0) \cdot e^{rt}$



" exponential growth function "

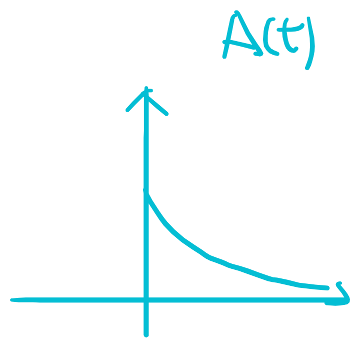
<u>Example</u>

① Radioactive decay.

Let $A(t)$ be the amount of a substance at time t, and k be the decay constant $(k<0)$

<u>Physical law</u>:

$$A'(t) = k \cdot A(t).$$



Math $\Rightarrow$

$$A(t) = A(0)\, e^{kt}$$

半衰期：

The number T with property

$$A(T) = \frac{1}{2} A(0)$$

半衰期
is called the "half-life" of this substance

$$A(T) = \cancel{A(0)}\, e^{kT} = \frac{1}{2}\cancel{A(0)}$$

$$\Rightarrow kT = \ln\left(\frac{1}{2}\right) = -\ln 2$$

$$\Rightarrow T = -\frac{\ln 2}{k} \quad \#$$

② Compound interest 複利

Let $A(t)$ be the amount of money at time t (year), and r be the annual interest. Then

$$A'(t) = r \cdot A(t)$$

$$\Rightarrow A(t) = A(0) \cdot e^{rt}$$

For example, (i) if

$$A(0) = 100000, \quad r = 1\%$$

$$\Rightarrow \quad A(t) = 100000 \cdot e^{0.01t}$$

$$\Rightarrow \quad A(10) \approx 110517$$

iii) If you need 150000 in 10 years,

then

$$A(10) = 10^5 \cdot e^{r \cdot 10} = 150000$$

$$\Rightarrow \quad r = \frac{1}{10} \ln\left(\frac{3}{2}\right) \approx 0.041$$

"4%"

<u>Verhulst</u>: logistic growth model

$$\boxed{\frac{dP}{dt} = k \cdot P(M-P)}$$

where M is the capacity of environment.

<u>Def</u> (9.2.1)

A differential equation is said to be separable if it can be

written as the form

written as the form

$$\boxed{p(x) + q(y)\cdot y' = 0}$$

$\int dx$

$$\Rightarrow \int p(x)\,dx + \int q(y)\cdot \underbrace{y'\,dx}_{dy} = C$$

$$= \int p(x)\,dx + \int q(y)\,dy = C$$

Example

① Solve

$$x + y\cdot y' = 0$$

sol

$$\int x + yy'\,dx = C$$

$$= \int x\,dx + \int y\,\boxed{y'dx}\;dy$$

$$= \frac{1}{2}x^2 + \frac{1}{2}y^2 = C$$

$$\Rightarrow \quad x^2+y^2 = C'$$

for some constant C' #

② Solve

$$\begin{cases} y + yy' = xy - y' & \text{— ⓐ} \\ y(2) = 1 & \text{— ⓑ} \end{cases}$$

sol

By ⓐ, $-xy + y + yy' + y' = 0$

$$= (1-x)y + (y+1)y' = 0$$

$$\Rightarrow \quad (1-x) + \frac{y+1}{y} y' = 0$$

$1 + \frac{1}{y}$ dy

$$\Rightarrow \quad \int 1-x \, dx + \int \frac{y+1}{y} y' dx = C$$

$$\Rightarrow \quad x - \frac{x^2}{2} + y + \ln|y| = C$$

By ⓑ (i.e. $y(2) = 1$),

By ...

$x=2$:

$$2-\frac{2^2}{2}+y(2)+\ln|y(2)| = C$$

$$= \cancel{2-2}+1+\overset{=0}{\cancel{\ln 1}} = C$$

$\Rightarrow \quad C=1$

So

$$x-\frac{x^2}{2}+y+\ln|y| = 1 \qquad \#$$

③ Solve Verhulst's population model:

(**) $$\frac{dP}{dt} = kP(M-P)$$

where $P=P(t)$ = polulation at time t

sol

By (**),

$$-k+\frac{1}{P(M-P)}\frac{dP}{dt} = 0$$

$\int dt$

$$\Rightarrow \quad -\int k\,dt+\int \frac{1}{P(M-P)}\frac{dP}{dt}\,dt = C$$

$$= -kt + \underbrace{\int \frac{1}{P(M-P)}\,dP}_{\textcircled{I}} = C$$

$\textcircled{I}$: Solve A, B in

$$\frac{1}{P(M-P)} \overset{+0P}{} = \frac{A}{P} + \frac{B}{M-P}$$

$$= \frac{A \cdot M + \overset{0}{(B-A)}P}{P(M-P)}$$

$$\Rightarrow \begin{cases} B-A=0 \\ A \cdot M = 1 \end{cases} \Rightarrow A = B = \frac{1}{M}$$

So

$$\int \frac{1}{P(M-P)}\,dP = \frac{1}{M}\int \frac{1}{P} + \frac{1}{M-P}\,dP$$

$$= \frac{1}{M}\left(\ln|P| - \ln|M-P|\right) + C'$$

$$= \underbrace{\frac{1}{M}\ln\left|\frac{P}{M-P}\right|}_{} + C'$$

Assume $0 < P < M$

Therefore,

$$-kt + \frac{1}{M}\ln\left|\overset{>0}{\frac{P}{M-P}}\right| = C$$

$$\Rightarrow \ln\left(\frac{P}{M-P}\right) = M(C+kt)$$

$$\Rightarrow \frac{P}{M-P} = e^{\ln\left(\frac{P}{M-P}\right)} = e^{MC+Mkt}$$

$$\Rightarrow P = M\cdot e^{MC+Mkt} - P\cdot e^{MC+Mkt}$$

$$\Rightarrow P(t) = \frac{M\cdot e^{MC+Mkt}}{1+e^{MC+Mkt}}$$

$$= \frac{\left(M\cdot \overbrace{e^{MC}}^{C''}\cdot e^{Mkt}\right)\cdot e^{-Mkt}}{\left(1+\underbrace{e^{MC}}_{C''}\cdot e^{Mkt}\right)\cdot e^{-Mkt}}$$

$$= \frac{M\cdot C''}{C''+e^{-Mkt}}$$

Assume $P(0) = R$, $\quad 0<R<M$.

Then

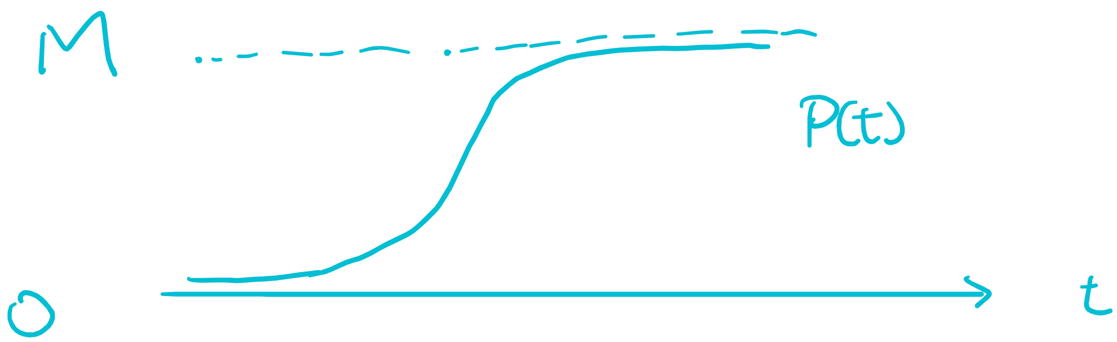
$$R = P(0) = \frac{M \cdot C''}{C'' + e^{-Mk \cdot 0}} = \frac{MC''}{C''+1}$$

$$\Rightarrow \quad C'' = \frac{R}{M-R} .$$

So

$$P(t) = \frac{MR}{R + (M-R) \cdot e^{-Mkt}}$$

which is called a logistic function. #

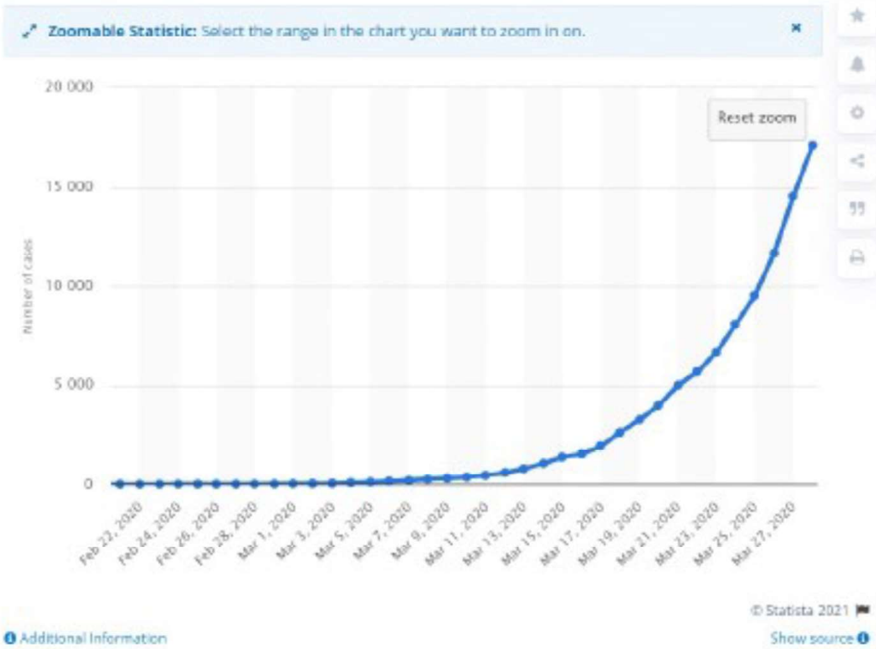


cited from statista.com

Health, Pharma & Medtech › State of Health

Cumulative number of coronavirus (COVID-19) cases in the United Kingdom (UK) since January 2020

(as of May 14, 2021)



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Source

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Release date

May 2021

Region

United Kingdom

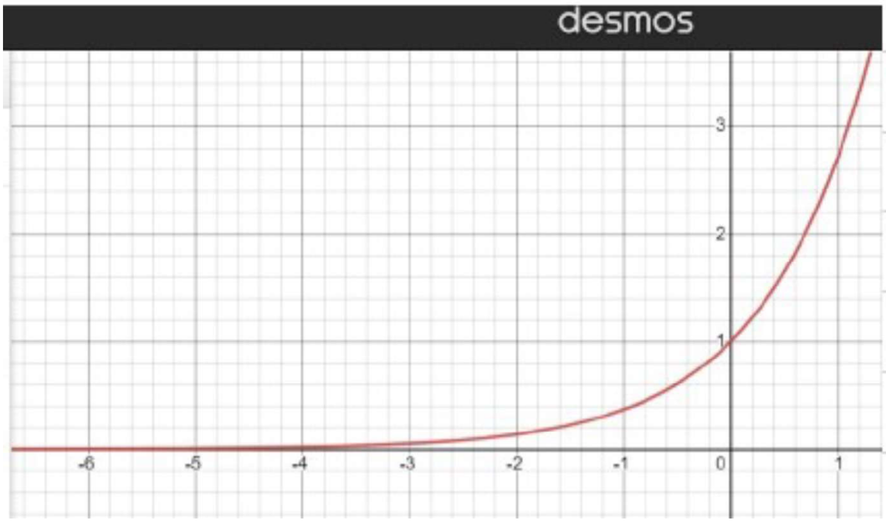
Survey time period

2021

Supplementary notes

*May 26: 2,039 extra confirmed cases have been added on this date due to historical data revisions according to the source.

** July 2: according to the source, due to methodological improvements and a revision to



← graph of $y = e^x$

from

desmos.com/calculator

1

$$\frac{5}{\left(1+4e^{-2x}\right)}$$

2