Calculus 12/28

Example

$(a+b)^2 = a^2+2ab+b^2$

① $$\int \frac{1}{x^2+x+1}\,dx = ?$$

x^2+x+1

$= \underbrace{x^2+2\cdot\frac{1}{2}\cdot x+\left(\frac{1}{2}\right)^2}_{(x+\frac{1}{2})^2} - \frac{1}{4}+1$

$= \left(x+\frac{1}{2}\right)^2+\frac{3}{4}$

$$= \int \frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\,dx$$

Recall

$\int \frac{1}{x^2+1}\,dx = \tan^{-1}x$

$$= \frac{4}{3}\int \frac{1}{\frac{4}{3}\left(x+\frac{1}{2}\right)^2+1}\,dx$$

Let

$u = \frac{2}{\sqrt{3}}x+\frac{1}{\sqrt{3}}$

$du = \frac{2}{\sqrt{3}}dx$

$$= \frac{4}{3}\int \frac{1}{\left(\frac{2}{\sqrt{3}}x+\frac{1}{\sqrt{3}}\right)^2+1}\,\boxed{dx}$$

$\boxed{dx} = \frac{\sqrt{3}}{2}du$

$$= \frac{2}{\sqrt{3}}\int \frac{1}{u^2+1}\,du = \frac{2}{\sqrt{3}}\tan^{-1}u + C$$

$$= \boxed{\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C} \quad \#$$

NOTE:

② $\int$ $x+\frac{1}{2}-\frac{1}{2}$

$(x^2+x+1)' = 2x+1$

$$\int \frac{\quad}{x^2+x+1}\,dx \qquad = 2\left(x+\tfrac{1}{2}\right)$$

$$= \int \frac{\overbrace{x+\frac{1}{2}}^{\frac{1}{2}du}}{x^2+x+1}\,dx - \frac{1}{2}\boxed{\int \frac{1}{x^2+x+1}\,dx}$$

Let $u = x^2+x+1$

$du = (2x+1)\,dx$

$$= \frac{1}{2}\overbrace{\boxed{\int \frac{1}{u}\,du}}^{\ln|u|} - \frac{1}{2}\int \frac{1}{x^2+x+1}\,dx$$

$$= \frac{1}{2}\ln|x^2+x+1| - \frac{1}{\cancel{2}}\,\frac{\cancel{2}^{\,1}}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C$$

#

③ $$\int \frac{1}{(x^2+x+1)^2}\,dx$$

$$= \int \frac{1}{\left((x+\frac{1}{2})^2+\frac{3}{4}\right)^2}\,dx$$

Let
$u = \dfrac{2x+1}{\sqrt{3}}$
$du = \dfrac{2}{\sqrt{3}}\,dx$

$$= \left(\frac{4}{3}\right)^2 \int \frac{1}{\left(\left(\frac{2x+1}{\sqrt{3}}\right)^2+1\right)^2}\,dx$$

$$\frac{16}{\quad}\ \sqrt{3}\int \frac{1}{\quad}$$

$$= \frac{16}{9}\cdot\frac{\sqrt{3}}{2}\int \frac{\quad}{(u^2+1)^2}\,du$$

$\tan^2\theta + 1 = \sec^2\theta$

Let

$u = \tan\theta$

$du = \sec^2\theta\, d\theta$

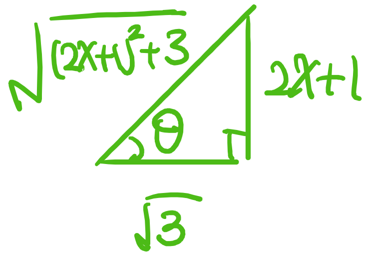
$$= \frac{8\sqrt{3}}{9}\int \frac{\sec^2\theta}{\boxed{\underbrace{(\tan^2\theta+1)}_{\sec^2\theta}{}^2}}\,d\theta$$

$\sec^4\theta$

$$= \frac{8\sqrt{3}}{9}\int \cos^2\theta\, d\theta$$

Recall:

$\frac{2x+1}{\sqrt{3}} = u = \tan\theta$



$$= \frac{8\sqrt{3}}{9}\int \frac{\cos 2\theta + 1}{2}\,d\theta$$

$$= \frac{8\sqrt{3}}{9}\left(\frac{\sin 2\theta}{4} + \frac{1}{2}\theta\right) + C$$

$\Rightarrow \sin\theta = \frac{2x+1}{\sqrt{4x^2+4x+4}}$

$\theta = \tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)$

$\cos\theta = \frac{\sqrt{3}}{\sqrt{4x^2+4x+4}}$

$$= \frac{8\sqrt{3}}{9}\left(\frac{\cancel{2}\sin\theta\cdot\cos\theta}{\cancel{4}\,2} + \frac{\theta}{2}\right) + C$$

$$= \frac{4\sqrt{3}}{9} \cdot \frac{\sqrt{3}(2x+1)}{4x^2+4x+4} + \frac{4\sqrt{3}}{9}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)$$

$$+ C \quad \#$$

④ $$\int \frac{x}{(x^2+x+1)^2}\,dx$$

idea:

Combine the methods of ② and ③

$$= \int \frac{x+\frac{1}{2}}{(x^2+x+1)^2}\,dx - \frac{1}{2}\int \frac{1}{(x^2+x+1)^2}\,dx$$

$\frac{1}{2}du$ ③

Let $u = x^2+x+1$

$\Rightarrow du = (2x+1)\,dx$

$$= \frac{1}{2}\int \frac{1}{u^2}\,du - \frac{1}{2}\int \frac{1}{(x^2+x+1)^2}\,dx$$

$$= \frac{1}{2}\left(\frac{-1}{u}\right) - \frac{1}{2}\int \frac{1}{(x^2+x+1)^2}\,dx$$

$$= -\frac{1}{2}\,\frac{1}{x^2+x+1} - \frac{1}{2}\left(\frac{1}{3}\cdot\frac{2x+1}{x^2+x+1} + \frac{4\sqrt{3}}{9}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)\right)$$

$+ C$ #

Differential equation

A differential equation is an equation which contains unknown functions and their derivatives.

For example,

$$y'(x) = y(x) \quad \leftarrow \text{order} = 1$$

and

$$y''(x) = x^2 + y(x) \quad \leftarrow \text{order} = 2$$

are differential equations.

The order of a differential equation the highest order of derivatives in the equation.

For example

1st order: $y'(x) = x$, $x \cdot y'(x) - y(x)$

2nd order: $y''(x) = x$, $x \cdot y''(x) - y'(x) = 0$

ODE

An ordinary differential equation (ODE) is a differential equation which consists of functions of one variable.

An partial differential equation (PDE) is a differential equation which consists of functions of many variables and partial derivatives.

Remark

The solution of the differential eq.

$$y'(x) = f(x)$$

are

$$\int f(x)\,dx = \int_0^x f(t)\,dt + C.$$

where C is an arbitrary constant.

A differential eq. may have infinitely many solutions.

The general solution of a differential equation consists of all the solutions of this equation

A particular solution of a differential equation is a certain speial solution of this equation.

3.a, 3.b (麟翔) 多人不知道怎麼使用微積分基本定理和連鎖律去解微分，並且有相當多人都不會對 $(\cos x)^{e^x}$ 微分。

3.d (麟翔) 很多人把 $(\ln x)^{\sin x}$ 看成 $\ln(x^{\sin x})$ 而寫錯。

4.b (俊碩) 有同學用變數變換 $u = 1 + t^2$，因在 $[-3, 3]$ 區間並不是一對一，所以儘管答案正確，依然沒有給分。

4.d (俊碩) 有少部分同學 $\sin x = e^{\ln(\sin x)}$。恰巧本題區間 sin 會落入負號。

5. (登科) 不少學生寫 $\cos^2 x$ 的反導函數是 $\frac{\cos^3 x}{3}$。

$y = \cos x$

$$\left(\frac{\cos^3 x}{3}\right)' = \frac{1}{3}\frac{dy^3}{dy} \cdot \frac{dy}{dx}$$

$$= \frac{1}{3} 3 \cdot y^2 \cdot (-\sin x)$$

$$= \cos^2 x \cdot (-\sin x) \neq \cos^2 x$$

National Tsing Hua University

Calculus I – Exam 2

Instructor: Hsuan-Yi Liao

Fall, 2023

Name: Answer

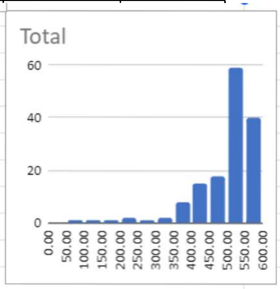
Student ID: ______________________

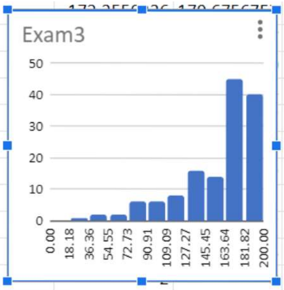
- This exam contains 9 pages (including this cover page) and 8 questions.
- Total of points is 200.
- Time limit: **100 minutes**.
- Write down your computation or arguments in details unless otherwise stated.
- The use of a calculator, cell phone, or any other electronic device is **NOT** permitted.
- The use of books or notes of any kind is **NOT** permitted.

Distribution of Marks

Question	Points	Score
1	10	
2	15	
3	32	
4	96	

Average	496.0067	157.9440559
Average (except zero)	496.0067	157.9440559
Quartile 0	93	25
Quartile 1	477.75	139.5
Quartile 2	521.5	170
Quartile 3	553.25	184
Quartile 4	587	200
標準差	85.6458	36.30248046
非零數	148	143
不到60%人數	8	20
滿分人數	0	3





1. (10 points) Find the critical points. Then find and classify all the extreme values.

$$f(x) = \begin{cases} x^2 + 2x + 2, & x < 0, \\ x^2 - 2x + 2, & 0 \le x \le 2. \end{cases}$$

See Exam 2

2. (15 points) Show that

$$\frac{1}{2} \le \int_1^2 \frac{1}{x}\,dx \le 1.$$

For $x \in [1, 2]$

$$\frac{1}{2} \le \frac{1}{x} \le 1$$

$$\Rightarrow \underbrace{\int_1^2 \frac{1}{2}\,dx}_{=\frac{1}{2}} \le \int_1^2 \frac{1}{x}\,dx \le \underbrace{\int_1^2 1\,dx}_{=1}$$

#

f: continuous

$\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x) \Rightarrow \frac{d}{dx}\left(\int_x^b f(t)\,dt\right) = \frac{d}{dx}\left(-\int_b^x f(t)\,dt\right) = -f(x)$

$= \frac{d}{dy}(\;)\cdot\frac{dy}{dx}$

3. Differentiate.

(a) (8 points) $\frac{d}{dx}\left(\int_{(\cos x)^{e^x}}^{1} (t-\ln t)\,dt\right)$, $0 < x < \frac{\pi}{2}$.

(b) (8 points) $\frac{d}{dx}\left(\int_{3x}^{1/x} \cos 2t\,dt\right)$, $x > 0$. $= \int_{3x}^{a} + \int_a^{\frac{1}{x}}$ $\frac{dy}{dx}$

$\left(e^{\ln(\cos x)^{e^x}}\right)' = \left(e^{e^x \ln\cos x}\right)' = e^{e^x \ln\cos x}\cdot\left(e^x \ln\cos x + e^x \frac{1}{\cos x}(-\sin x)\right) = (\cos x)^{e^x}\cdot\left(e^x \ln\cos x - e^x \tan x\right)$

$y - \ln y$

(a) $= -\left((\cos x)^{e^x} - \ln(\cos x)^{e^x}\right)\cdot\left((\cos x)^{e^x}\right)'$

$= \left(e^x\cdot\ln(\cos x) - (\cos x)^{e^x}\right)\cdot(\cos x)^{e^x}\cdot\left(e^x\ln(\cos x) - e^x\tan x\right)$ #

(b) $= \left(\cos\frac{2}{x}\right)\cdot\left(\frac{1}{x}\right)' - \cos(2\cdot 3x)\cdot(3x)'$

$= -\frac{1}{x^2}\cos\left(\frac{2}{x}\right) - 3\cos 6x$ #

$\left((\cos x)^{e^x}\right)' = \left(e^{\ln(\cos x)^{e^x}}\right)' = \left(e^{e^x\cdot\ln(\cos x)}\right)'$

$= \underbrace{e^{e^x\cdot\ln(\cos x)}}_{=(\cos x)^{e^x}}\cdot\left(e^x\cdot\ln\cos x\right)'$ $\quad \frac{(\cos x)'}{\cos x} = -\frac{\sin x}{\cos x}$

$= (\cos x)^{e^x}\cdot\left((e^x)'\cdot\ln\cos x + e^x\cdot(\ln\cos x)'\right)$, $(e^x)' = e^x$

$= (\cos x)^{e^x}\cdot\left(e^x\ln(\cos x) - e^x\cdot\tan x\right)$

(c) (8 points) $\frac{d}{dx}(x^x)$, $\quad x > 0$.

(d) (8 points) $\frac{d}{dx}\left((\ln x)^{\sin x}\right)$, $\quad x > 0$.

$$(c) = \left(e^{\ln x^x}\right)' = \left(e^{x\ln x}\right)' = e^{x\ln x}\left(\ln x + x \cdot \frac{1}{x}\right)$$

$$= x^x(\ln x + 1) \quad \#$$

$$(d) = \left(e^{\ln(\ln x)\cdot \sin x}\right)' = e^{\sin x \cdot \ln(\ln x)} \cdot \left(\cos x \cdot \ln(\ln x) + \sin x \frac{1}{\ln x} \cdot \frac{1}{x}\right)$$

$$= (\ln x)^{\sin x} \cdot \left(\cos x \cdot \ln(\ln x) + \frac{\sin x}{x \ln x}\right) \quad \#$$

4. Integrate.

(a) (8 points) $\int_1^2 (2x + x^2)\,dx.$ $= x^2+\frac{x^3}{3}\Big|_1^2 = (4+\frac{8}{3}-1-\frac{1}{3}) = 3+\frac{7}{3} = \frac{16}{3}$ #

(b) (8 points) $\int_{-3}^{3} \frac{t^3}{1+t^2}\,dt.$ $= 0$ $(\because \frac{t^3}{1+t^2}$ is odd $)$

(c) (8 points) $\int_2^5 |x-3|\,dx.$ $= \int_2^3 3-x\,dx + \int_3^5 x-3\,dx = 3x-\frac{x^2}{2}\Big|_2^3 + \frac{x^2}{2}-3x\Big|_3^5$

$= 3-\frac{9}{2}+\frac{4}{2}+\frac{25}{2}-\frac{9}{2}-6 = \frac{5}{2}$ #

(d) (8 points) $\int_0^\pi e^x \sin x\,dx.$

(d) $\boxed{\int_0^\pi e^x \sin x\,dx} = \cancel{e^x \sin x\Big|_0^\pi}^{\,=0} - \int_0^\pi e^x \cos x\,dx$ $\quad v = e^x \quad u = \sin x$ $\quad v' = e^x \quad u' = \cos x$

$\int u v' dx = uv - \int v \cdot u' dx$

$= \underbrace{-e^x \cos x\Big|_0^\pi}_{= e^\pi + 1} + \boxed{\int_0^\pi e^x (-\sin x)\,dx}$

$\Rightarrow \int_0^\pi e^x \sin x\,dx = \frac{1}{2}(e^\pi + 1)$ #

$v = e^x \quad u = \cos x$

$v' = e^x \quad u' = -\sin x$

$u = \sin(\frac{\pi}{2}x) \Rightarrow du = \frac{\pi}{2}\cos(\frac{\pi}{2}x)\,dx$

(e) (8 points) $\int_0^1 \cos(\frac{\pi}{2}x)\sin^2(\frac{\pi}{2}x)\,dx.$ $= \int_0^1 \frac{2}{\pi}\, u^2 \, du = \frac{2}{\pi}\frac{u^3}{3}\Big|_0^1 = \frac{2}{3\pi}$ #

(f) (8 points) $\int_0^1 x(x+1)^9\,dx.$ $= x\frac{(x+1)^{10}}{10}\Big|_0^1 - \int_0^1 \frac{(x+1)^{10}}{10}dx = \frac{2^{10}}{10} - \frac{(x+1)^{11}}{10\cdot 11}\Big|_0^1$

(g) (8 points) $\int_0^1 \frac{x}{\sqrt{1+x}}\,dx.$ $= \frac{2^{10}}{10} - \frac{2^{11}}{110} + \frac{1}{110}$ #

(h) (8 points) $\int_{e^3}^{e} \frac{\ln x}{x}\,dx.$

(g) $= \int_0^1 x\,(1+x)^{-\frac{1}{2}}\,dx = x\cdot 2(1+x)^{\frac{1}{2}}\Big|_0^1 - \int_0^1 2(1+x)^{\frac{1}{2}}\,dx$

$= 2\sqrt{2} - 2\cdot\frac{2}{3}(1+x)^{\frac{3}{2}}\Big|_0^1 = 2\sqrt{2} - \frac{8}{3}\sqrt{2} + \frac{4}{3}$

$= \frac{4}{3} - \frac{2}{3}\sqrt{2}$ #

$u = \ln x$

(h) $= \int_3^1 u\,du = \frac{u^2}{2}\Big|_3^1 = \frac{1}{2} - \frac{9}{2} = -4$ #

$du = \frac{1}{x}dx$

(i) (8 points) $\displaystyle\int_1^2 \frac{1}{\sqrt{x}(1+\sqrt{x})}\,dx.$ $= \displaystyle\int_2^{1+\sqrt{2}} \frac{1}{u}\, 2du = 2\cdot\ln\left(\frac{1+\sqrt{2}}{2}\right)$ #

$u = 1+\sqrt{x}$

$du = \frac{1}{2}\frac{1}{\sqrt{x}}dx$

(j) (8 points) $\displaystyle\int_0^{\ln\pi} e^{-2x}\,dx.$ $= \displaystyle\frac{-1}{2} e^{-2x}\Big|_0^{\ln\pi} = -\frac{1}{2}\pi^{-2} + \frac{1}{2}$ #

(k) (8 points) $\displaystyle\int_0^1 \frac{e^x}{4-e^x}\,dx.$

(l) (8 points) $\displaystyle\int_1^2 2^{-x}\,dx.$

$u = 4-e^x$

(k) $= \displaystyle\int_3^{4-e} \frac{-1}{u}\,du = -\ln(4-e) + \ln 3$ #

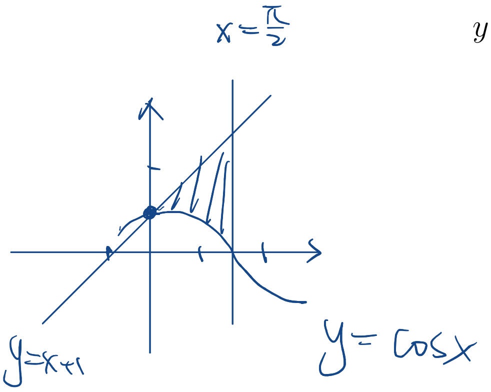
$du = -e^x dx$

(l) $= \displaystyle -\frac{2^{-x}}{\ln 2}\Big|_1^2 = -\frac{\frac{1}{4}}{\ln 2} + \frac{\frac{1}{2}}{\ln 2}$

$= \displaystyle\frac{1}{4\ln 2}$ #

5. (15 points) Sketch the region Ω bounded by the curves and find the volume of the solid generated by revolving this region about the x-axis.

$$y = \cos x, \qquad y = x + 1, \qquad x = \frac{1}{2}\pi.$$



$$\boxed{\frac{\cos 2x + 1}{2}} = $$

$$\text{Vol} = \int_0^{\frac{\pi}{2}} \pi\left((x+1)^2 - \boxed{(\cos x)^2}\right) dx$$

$$= \pi\left(\frac{(x+1)^3}{3} - \frac{\frac{1}{2}\sin 2x + x}{2}\right)\Bigg|_0^{\frac{\pi}{2}}$$

$$= \pi \frac{(\frac{\pi}{2}+1)^3}{3} - \frac{\pi}{3} - \pi\frac{\pi}{4}$$

$$= \frac{\pi}{3}\left(\frac{\pi^3}{8} + \frac{3}{4}\pi^2 + \frac{3}{2}\pi + \cancel{1}\right) - \cancel{\frac{\pi}{3}} - \frac{\pi^2}{4}$$

$$= \frac{\pi^4}{24} + \frac{\pi^3}{4} + \frac{\pi^2}{4} \quad \#$$

6. Set

$$f(x) = \int_2^x \sqrt{1+t^2}\, dt.$$

(a) (5 points) Show that f is one-to-one.

(b) (5 points) Find $(f^{-1})'(0)$.

(a) Since $f'(x) = \sqrt{1+x^2} > 0 \quad \forall x \in \mathbb{R}$,
f is strictly increasing $\Rightarrow$ 1-1 #

(b) $f(2) = 0$

$$(f^{-1})'(0) = \frac{1}{f'(2)} = \frac{1}{\sqrt{1+2^2}} = \frac{1}{\sqrt{5}} \quad \#$$

7. (10 points) Show that $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$. (Assume $x>0$)

Since for $t \in [1, 1+\frac{1}{x}]$,

$$\frac{x}{x+1} = \frac{1}{1+\frac{1}{x}} \leq \frac{1}{t} \leq 1$$

we have

$$\int_1^{1+\frac{1}{x}} \frac{x}{x+1}\,dx \leq \ln\left(1+\frac{1}{x}\right) = \int_1^{1+\frac{1}{x}} \frac{1}{t}\,dt \leq \int_1^{1+\frac{1}{x}} 1\,dx = \frac{1}{x}$$

$$\int_1^{1+\frac{1}{x}} \frac{x}{x+1}\,dx = \frac{1}{x+1}$$

$$\Rightarrow \underbrace{\frac{x}{x+1}}_{\to 1 \text{ as } x\to\infty} \leq x\ln\left(1+\frac{1}{x}\right) \leq 1.$$

$$\Rightarrow \lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right) = 1 \quad \text{by Pinching Thm}$$

$$\Rightarrow \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty} e^{x\ln\left(1+\frac{1}{x}\right)} = e^{\lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right)}$$

$$= e^1 = e \quad \#$$

8. True or false. No need to explain. Assume f and g are continuous on $[a,b]$, $a<b$.

T (a) (2 points) If $\int_a^b f(x)\,dx > \int_a^b g(x)\,dx$, then $\int_a^b [f(x)-g(x)]\,dx > 0$.

F (b) (2 points) If $\int_a^b f(x)\,dx > \int_a^b g(x)\,dx$, then $f(x) > g(x)$ for all $x \in [a,b]$.

F (c) (2 points) If $\int_a^b f(x)\,dx = 0$, then $f(x) = 0$ for all $x \in [a,b]$.

T (d) (2 points) If $\int_a^b f(x)\,dx = 0$, then $f(x) = 0$ for some $x \in [a,b]$.

T (e) (2 points) If $\int_a^b |f(x)|\,dx = 0$, then $f(x) = 0$ for all $x \in [a,b]$.

F (f) (2 points) If $\int_a^b f(x)\,dx = 0$, then $\int_a^b |f(x)|\,dx = 0$.