Calculus 12/26

Example

①

$$\int \frac{1}{\sqrt{x^2+2x+5}}\,dx = ?$$

$2\cdot1\cdot x$ $\quad$ $1+4$ $\quad$ $(a+b)^2 = a^2+2ab+b^2$

$$= \int \frac{1}{\sqrt{(x+1)^2+2^2}}\,dx$$

Let $y = x+1$

$dy = dx$

$$= \int \frac{1}{\sqrt{y^2+2^2}}\,dy$$

Let $x+1 = y = 2\tan u$

$dy = 2\sec^2 u\,du$

$$= \int \frac{1}{\sqrt{2^2\tan^2 u+2^2}}\,2\cdot\sec^2 u\,du$$

$2^2(\tan^2 u+1)$

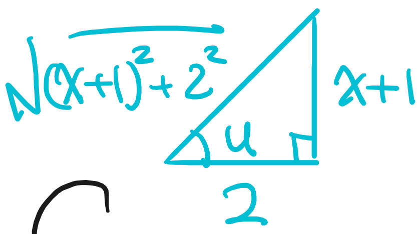
$= 2^2\sec^2 u$

$$= \int \frac{1}{\cancel{2}\,\cancel{\sec u}}\,\cancel{2}\sec^{\cancel{2}} u\,du$$

it's $|\sec u|$ to be

precise

$$= \int \sec u \, du \qquad \tan u = \frac{x+1}{2}$$



$$= \ln\left|\sec u + \tan u\right| + C$$

$$= \ln\left|\frac{\sqrt{(x+1)^2+2^2}}{2} + \frac{x+1}{2}\right| + C$$

$$= \ln\left|\sqrt{x^2+2x+5} + x+1\right| \overbrace{-\ln 2 + C}^{C'}$$

$$= \ln\left|\sqrt{x^2+2x+5} + x+1\right| + C' \quad \#$$

<u>Rational function</u> (§ 8.5)

Let $P(x)$, $Q(x)$ be polynomials.

Q: $\int \frac{P(x)}{\ldots} \, dx = ?$

$$\int \frac{\quad}{Q(x)}\, dx = ?$$

Step 1. If $\deg P(x) \geq \deg Q(x)$, then use the division algorithm to get $g(x)$, $r(x)$ s.t.

$$P(x) = g(x) \cdot Q(x) + r(x)$$

$$\Rightarrow \quad \frac{P(x)}{Q(x)} = g(x) + \frac{r(x)}{Q(x)}$$

$\deg r(x) < \deg Q(x)$

Example

Let $P(x) = x^3 + x^2 + x + 1$

$Q(x) = x - 1$

Q: $$\int \frac{x^3 + x^2 + x + 1}{x - 1}\, dx = ?$$

sol

$$x^2 + 2x + 3$$

$$
\begin{array}{r|l}
x-1 & x^3+x^2+x+1 \\
 & \underline{x^3-x^2} \\
 & \quad 2x^2+x \\
 & \quad \underline{2x^2-2x} \\
 & \qquad 3x+1 \\
 & \qquad \underline{3x-3} \\
 & \qquad\quad 4
\end{array}
$$

$$\Rightarrow \quad x^3+x^2+x+1 = (x^2+2x+3)(x-1)+4$$

$$\Rightarrow \quad \frac{x^3+x^2+x+1}{x-1} = x^2+2x+3 + \frac{4}{x-1}$$

$$\Rightarrow \quad \int \frac{x^3+x^2+x+1}{x-1}\,dx = \int x^2+2x+3+\frac{4}{x-1}\,dx$$

$$= \frac{x^3}{3} + x^2 + 3x + 4\ln|x-1| + C$$

#

Step2. By Step1, we can assume

$\deg P(x) < \deg Q(x)$.

To compute

$$\int \frac{P(x)}{Q(x)}\,dx ,$$

因式分解

we factorize $Q(x)$ and

decompose $\dfrac{P(x)}{Q(x)}$

Example

$$\int \frac{\overset{=P(x)}{1}}{\underbrace{x^2-2x-3}_{=Q(x)}}\,dx = ?$$

Sol

$(-3)\cdot(+1)$

$$x^2-2x-\underline{\underline{3}} = (x-3)(x+1)$$

Assume

$$\frac{P(x)}{Q(x)} = \frac{A}{x-3} + \frac{B}{x+1}$$

Solve A, B:

$$\frac{0\cdot x+1}{(x-3)(x+1)} = \frac{A(x+1)+B(x-3)}{(x-3)(x+1)}$$

$$= \frac{(A+B)x + (A-3B)}{(x-3)(x+1)}$$

$$\Rightarrow \begin{cases} A+B=0 \Rightarrow A=-B \\ A-3B=1 \Rightarrow -4B=1 \Rightarrow B=-\frac{1}{4} \end{cases} \qquad A=\frac{1}{4}$$

So $$\frac{1}{(x-3)(x+1)} = \frac{1}{4}\cdot\frac{1}{x-3} - \frac{1}{4}\cdot\frac{1}{x+1}$$

$$\Rightarrow \int \frac{1}{(x-3)(x+1)}\,dx = \int \frac{1}{4}\cdot\frac{1}{x-3} - \frac{1}{4}\cdot\frac{1}{x+1}\,dx$$

$$= \frac{1}{4}\int\frac{1}{x-3}\,dx - \frac{1}{4}\int\frac{1}{x+1}\,dx$$

$$= \frac{1}{4}\ln|x-3| - \frac{1}{4}\ln|x+1| + C$$

$$= \frac{1}{4}\ln\left|\frac{x-3}{x+1}\right| + C \qquad \#$$

Remark

One can factorize a real polynomial:

$$Q(x) = a \cdot (x-b_1)^{r_1} \cdot (x-b_2)^{r_2} \cdots (x-b_i)^{r_i} \cdot (x^2+c_{i+1}x+d_{i+1})^{r_{i+1}} \cdots (x^2+c_j x+d_j)^{r_j}$$

$$\Rightarrow \frac{P(x)}{Q(x)} = a\left(\frac{f_1(x)}{(x-b_1)^{r_1}} + \cdots + \frac{f_i(x)}{(x-b_i)^{r_i}} + \frac{f_{i+1}(x)}{(x^2+c_{i+1}x+d_{i+1})^{r_{i+1}}} + \cdots + \frac{f_j(x)}{(x^2+c_j x+d_j)^{r_j}} \right)$$

We shall compute

case1 $\displaystyle\int \frac{f(x)}{(x-b)^r}\,dx$ and case2 $\displaystyle\int \frac{f(x)}{(x^2+cx+d)^r}\,dx$

Example

① (case 1)

$$\int \frac{x+1}{(x-1)^2}\,dx = ?$$

idea: Decompose

$$\frac{x+1}{(x-1)^2} = \frac{A}{(x-1)^2} + \frac{B}{x-1}$$

sol

Solve A, B in

$$\frac{\boxed{1}x+\boxed{1}}{(x-1)^2} = \frac{A}{(x-1)^2} + \frac{B(x-1)}{(x-1)^2}$$

$$= \frac{\boxed{B}x + \boxed{(A-B)}}{(x-1)^2}$$

$$\Rightarrow \begin{cases} A-B=1 \\ B=1 \end{cases} \Rightarrow \begin{cases} A=2 \\ B=1 \end{cases}$$

$2(x-1)^{-2} = (-2(x-1)^{-1})'$

So $\int \frac{x+1}{(x-1)^2}dx = \int \frac{2}{(x-1)^2} + \frac{1}{x-1}\, dx$

$$= \frac{-2}{x-1} + \ln|x-1| + C \ \#$$

② (case 2-1)

$$\int \frac{x^3}{(x^2+1)^2}dx = ?$$

deg = 3 > 2

idea

$$\frac{x^3}{(x^2+1)^2} = \frac{Ax+B}{(x^2+1)^2} + \frac{Cx+D}{x^2+1}$$

sol

$Cx^3 + Dx^2 + Cx + D$

Solve A, B, C, D

$$\frac{\boxed{1}x^3}{(x^2+1)^2} = \frac{Ax+B}{(x^2+1)^2} + \frac{(Cx+D)(x^2+1)}{(x^2+1)^2}$$

$$= \frac{\boxed{C}x^3 + \overset{\overset{0}{\shortparallel}}{D}x^2 + \overset{\overset{0}{\shortparallel}}{(A+C)}x + \overset{\overset{0}{\shortparallel}}{(B+D)}}{(x^2+1)^2}$$

$\Rightarrow \quad C=1, \quad D=0, \quad A+C=0, \quad B+D=0$

$\Rightarrow \quad A=-1, \quad B=0, \quad C=1, \quad D=0$

So

$$\int \frac{x^3}{(x^2+1)^2}\,dx = \int \frac{-x}{(x^2+1)^2} + \overbrace{\frac{x}{x^2+1}\,dx}^{\frac{1}{2}du}$$

Observation: $(x^2+1)' = 2x$

Let $u = x^2+1 \Rightarrow du = 2x\,dx$

$$= \int \underbrace{\frac{-\frac{1}{2}}{u^2}}_{\left(\frac{1}{2}u^{-1}\right)'}du + \int \overbrace{\frac{1}{u}\,\frac{1}{2}}^{\left(\frac{1}{2}\ln|u|\right)'}du$$

$$= \frac{1}{2}\,\frac{1}{u} + \frac{1}{2}\ln|u| + C$$

$$= \frac{1}{2}\frac{1}{x^2+1} + \frac{1}{2}\ln|x^2+1| + C \#$$

③ (case 2-2)

$$\int \frac{1}{x^2+x+1}\,dx$$

idea:

$x^2+x+1 = (x+a)^2+b$

use $\int \frac{1}{x^2+1}dx = \tan^{-1}x$

sol