Calculus 12/21

Q:

$$\int \sin^p x \cos^q x \, dx = ?$$

<u>case1</u>: p or q is odd.

Assume $p = 2k+1$.

$\sin^2 x + \cos^2 x = 1$
$\Rightarrow \sin^2 x = 1 - \cos^2 x$

$$\int \sin^p x \cos^q x \, dx = \int \underline{\sin^{2k} x} \cdot \sin x \cos^q x \, dx$$

$$= \int (1-\cos^2 x)^k \cdot \cos^q x \cdot \underbrace{\sin x \, dx}_{-du}$$

Let $u = \cos x \Rightarrow du = -\sin x \, dx$

$$= -\int u^q (1-u^2)^k \, du$$

<u>Example</u>

$(\cos^2 x)^2 \cdot \cos x = (1-\sin^2 x)^2 \cos x$

① $\int \sin^2 x \cdot \boxed{\cos^5 x} \, dx$

Let $u = \sin x$

$\int \sin^2 x \, \boxed{\cos x} \, dx$

$\Rightarrow du = \cos x \, dx$

$$= \int \sin^2 x \cdot (1-\sin^2 x)^2 \cdot \underbrace{\cos x \, dx}_{du}$$

$1-2u^2+u^4$

$$= \int u^2 (1-u^2)^2 \, du$$

$$= \int u^2 - 2u^4 + u^6 \, du$$

$$= \frac{u^3}{3} - \frac{2}{5}u^5 + \frac{u^7}{7} + C$$

$$= \frac{\sin^3 x}{3} - \frac{2}{5}\sin^5 x + \frac{\sin^7 x}{7} + C$$

#

$(\sin^2 x)^2 \cdot \sin x$

② $\int \sin^5 x \, dx = \int (1-\cos^2 x)^2 \cdot \underbrace{\sin x \, dx}_{-du}$

Let $u = \cos x$

$du = -\sin x \, dx$

$$= -\int (1-u^2)^2 \, du$$

$$= -\int 1 - 2u^2 + u^4 \, du$$

$$= -u + \frac{2}{3}u^3 - \frac{u^5}{5} + C$$

$$= -\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + C \quad \#$$

Case 2: Both p and q are even.

Then use the product-to-sum formulas to reduce the powers.

$$\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x = 2\cos^2 x - 1$$

Example

$$\int \sin^2 x \cos^6 x \, dx = \int \underbrace{(\sin x \cdot \cos x)^2}_{} \, \underbrace{(\cos^2 x)^2}_{} dx$$

$$\sin x \cdot \cos x = \frac{1}{2}\sin 2x \qquad \cos^2 x = \frac{\cos 2x + 1}{2}$$

$$= \frac{1}{16}\int (\sin 2x)^2 \cdot (\cos 2x + 1)^2 \, dx$$

$$(\sin 2x \cdot \cos 2x)^2 = \left(\frac{\sin 4x}{2}\right)^2$$

$$= \frac{1}{16}\int \sin^2 2x \cdot \cos^2 2x + 2\sin^2 2x \cdot \cos 2x + \sin^2 2x \, dx$$

$$\sin 2x \cdot \underline{\sin 2x \cdot \cos 2x} \qquad \frac{1-\cos 4x}{2}$$

$$= \sin 2x \cdot \frac{\sin 4x}{2}$$

$$= \frac{1}{64}\int \underbrace{\sin^2 4x}_{\frac{1-\cos 8x}{2}}\, dx + \frac{1}{16}\int \underbrace{\sin 2x \sin 4x}_{\cos(2x-4x) - \cos(2x+4x)}\, dx$$

$$+ \frac{1}{32}\int 1 - \cos 4x \, dx$$

$$\left(\frac{\sin 8x}{8}\right)'$$

$$= \frac{1}{128}\int 1 - \cos 8x \, dx + \frac{1}{16}\int \cos 2x - \cos 6x \, dx$$

$$+ \frac{1}{32}\int 1 - \cos 4x \, dx$$

$$= \frac{x}{128} - \frac{\sin 8x}{8\times 128} + \frac{\sin 2x}{16\times 2} - \frac{\sin 6x}{16\times 6}$$

$$+ \frac{x}{32} - \frac{\sin 4x}{32\times 4} + C \quad \#$$

Trigonometric substitution

Q: Integrate

$\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$, $\sqrt{x^2-a^2}$

$(a>0)$

Recall

(i) $\cos^2 x + \sin^2 x = 1$

(ii) $1 + \tan^2 x = \sec^2 x$ (§8.4)

Principle:

(i) type $\sqrt{a^2-x^2}$: let $x = a\sin u$ ($\sqrt{a^2-x^2} \to a\cos u$)

(ii) " $\sqrt{a^2+x^2}$: let $x = a\tan u$ ($\sqrt{a^2+x^2} \to a\sec u$)

(iii) " $\sqrt{x^2-a^2}$: let $x = a\sec u$ ($\sqrt{x^2-a^2} \to a\tan u$)

Example

① (type $\sqrt{a^2-x^2}$)

$$\int_{-a}^{a} \sqrt{a^2-x^2}\,dx$$

Let $x = a\sin u$

$dx = a\cos u\,du$

$x=a$:

$a = a\sin u$

$\Rightarrow \sin u = 1$

$\Rightarrow u = \sin^{-1} 1$

$a\cos u$

$\|$

$\int^{u(a)} \; u(a) = \sin^{-1} 1 = \frac{\pi}{2}$

$$= \int_{u(-a)}^{} \underbrace{\sqrt{a^2 - (a\sin u)^2}} \quad a\cos u \, du$$

$u(-a) = \sin^{-1} -1 = -\frac{\pi}{2}$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} a^2 \underbrace{\cos^2 u}_{} \, du \qquad = \frac{1+\cos 2u}{2}$$

$$= a^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1+\cos 2u}{2} \, du$$

$$= a^2 \left(\frac{1}{2}u + \frac{\sin 2u}{2\times 2} \right) \Bigg|_{u=-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= a^2 \frac{\pi}{2} \qquad \#$$

② (type $\sqrt{a^2+x^2}$)

Let $x = a\tan u$

$dx = a\sec^2 u \, du$

$$\int \sqrt{a^2+x^2} \, dx$$

(It's $a|\sec u|$ to be precise)

$a \cdot \sec u$

$\because 1+\tan^2 u = \sec^2 u$

$$= \int \sqrt{a^2 \quad \ldots} \quad a\sec^2 u \, du$$

$$= \int \sqrt{a^2 + a^2 \tan^2 u} \quad a \cdot \sec^2 u \, du$$

$$= a^2 \int \sec^3 u \, du$$

Q: $\int \sec^3 u \, du = \, ?$ $\quad = \sec u \cdot \tan u$

$$\left(\frac{1}{\cos u}\right)' = -\frac{(\cos u)'}{\cos^2 u} = \frac{\sin u}{\cos^2 u}$$

$$\int \sec u \cdot \underbrace{\sec^2 u}_{= (\tan u)'} \, du$$

integration by parts

$$= \sec u \cdot \tan u - \int \underbrace{(\sec u)' \cdot \tan u}_{= \sec u \cdot \tan^2 u} \, dx$$

$$\sec u \cdot \tan^2 u = \sec u (\sec^2 u - 1) = \sec^3 u - \sec u$$

$$= \sec u \cdot \tan u - \int \sec^3 u \, du + \int \sec u \, du$$

$$\Rightarrow \int \sec^3 u \, du = \frac{1}{2}\left(\sec u \cdot \tan u + \int \sec u \, du\right)$$

Recall:

$$\int \sec u \, du = \ln|\sec u + \tan u| + C$$

pf

$$(\ln|\sec u + \tan u|)'$$

$$= \frac{1}{\sec u + \tan u}(\sec u + \tan u)'$$

$$= \frac{1}{\sec u + \tan u}\underbrace{(\sec u \cdot \tan u + \sec^2 u)}_{\sec u \cdot (\tan u + \sec u)}$$

$$= \sec u \quad \#$$

So

$$\int \sec^3 u \, du = \frac{1}{2}\left(\sec u \cdot \tan u + \int \sec u \, du\right)$$

$$= \frac{1}{2}\left(\sec u \cdot \tan u + \ln|\sec u + \tan u|\right) + C$$

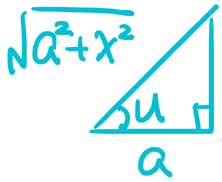
So

$$\int \sqrt{a^2 + x^2} \, dx = a^2 \int \sec^3 u \, du$$

$$= \frac{a^2}{2}\left(\sec u \cdot \underbrace{\tan u}_{=\frac{x}{a}} + \ln\left|\sec u + \underbrace{\tan u}_{=\frac{x}{a}}\right|\right) + C$$

Replace u by x:

Recall $x = a\tan u$



$\Rightarrow \sec u = \frac{\sqrt{a^2+x^2}}{a}$

$$\Rightarrow \int \sqrt{a^2+x^2}\,dx$$

$$= \frac{a^2}{2}\left(\frac{\sqrt{a^2+x^2}}{a}\cdot\frac{x}{a} + \ln\left|\overbrace{\frac{\sqrt{a^2+x^2}}{a}+\frac{x}{a}}^{\frac{x+\sqrt{a^2+x^2}}{a}}\right|\right)+C$$

$$= \frac{x\sqrt{a^2+x^2}}{2} + \frac{a^2}{2}\left(\ln\left|x+\sqrt{a^2+x^2}\right|\right)\underbrace{\overbrace{-\frac{a^2}{2}\ln a}^{\text{constant}} + C}_{C'}$$

$$= \frac{x\sqrt{a^2+x^2}}{2} + \frac{a^2}{2}\ln\left|x+\sqrt{a^2+x^2}\right| + C' \quad \#$$

③ (type $\sqrt{x^2-a^2}$)

Let $x = \sec u$

$\Rightarrow dx = \tan u \sec u\, du$

$$\int \frac{1}{\sqrt{x^2-1}}\,dx$$

$\Rightarrow \tan u$

$u \qquad 1 \qquad = \sqrt{x^2-1}$

$$= \int \frac{1}{\sqrt{\sec^2 u - 1}} \cancel{\tan u} \sec u \, du$$

$\sqrt{\sec^2 u - 1} = \cancel{\tan u} \leftarrow$ (it's $|\tan u|$ to be precise)

$$= \int \sec u \, du = \ln|\sec u + \tan u| + C$$

$$= \ln\left| x + \sqrt{x^2-1} \right| + C \quad \#$$