Calculus 13/9

<u>Recall</u>

① $(\sin^{-1} x)' = \dfrac{1}{\sqrt{1-x^2}}$, i.e.

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$$

② $(\tan^{-1} x)' = \dfrac{1}{1+x^2}$, i.e.

$$\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C$$

Furthermore, by a similar method

- $\cos^{-1} : [-1, 1] \to [0, \pi]$,

  $$(\cos^{-1} x)' = \frac{-1}{\sqrt{1-x^2}}$$

- $\sec^{-1} : (-\infty, -1] \cup [1, \infty) \to [0\ \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$

  $$(\sec^{-1} x)' = \frac{1}{}$$

$(\sec^{-1}x)' = \frac{\quad}{|x|\sqrt{x^2-1}}$

- $\cot^{-1}: \mathbb{R} \to (0,\pi)$

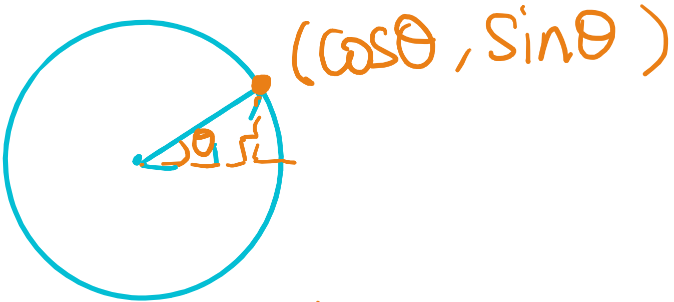
  $(\cot^{-1}x)' = \dfrac{-1}{1+x^2}$

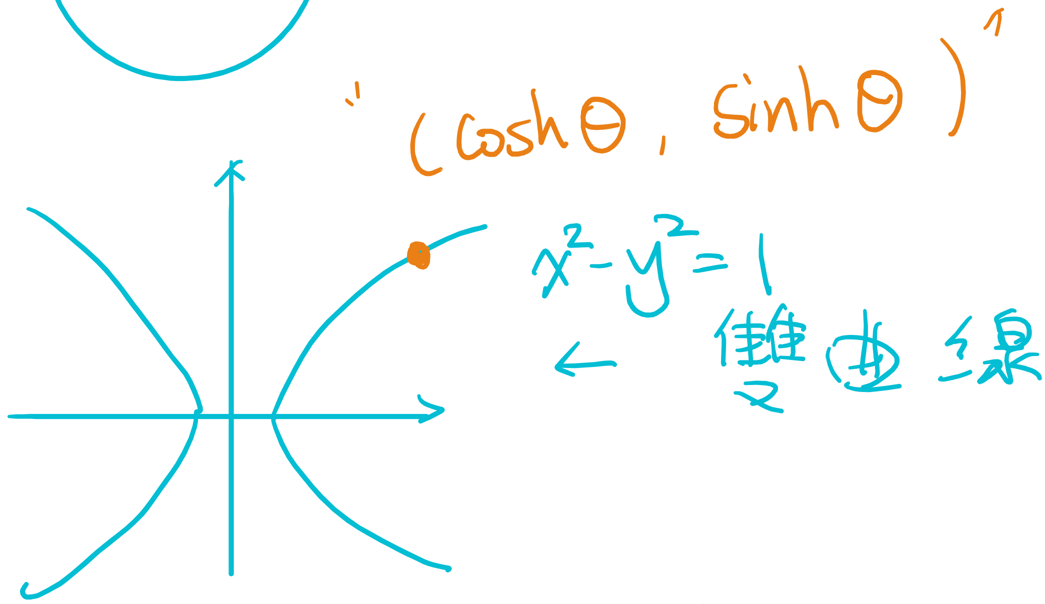
- $\csc^{-1}: (-\infty,-1] \cup [1,\infty) \to [-\frac{\pi}{2},0) \cup (0,\frac{\pi}{2}]$

  $(\csc^{-1}x)' = \dfrac{-1}{|x|\sqrt{x^2-1}}$

雙曲線的

Hyperbolic sine and cosine





$\sinh(-x) = -\sinh x$

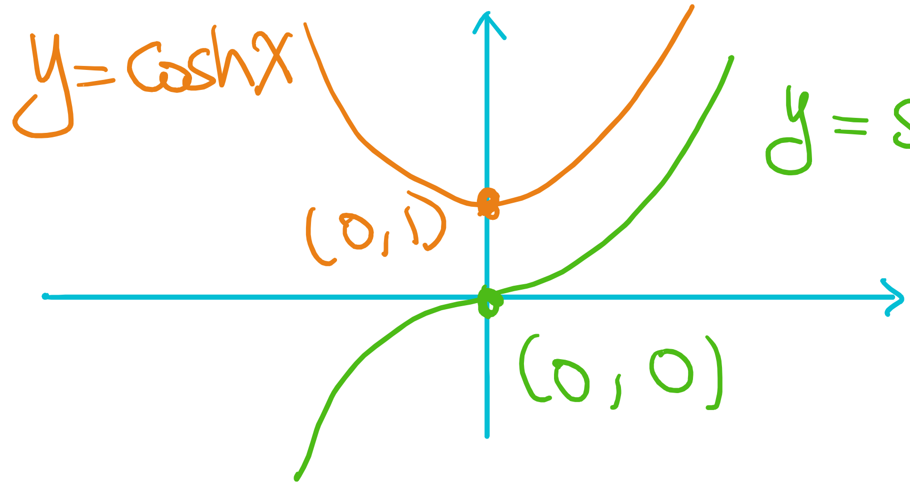
Def

odd

$$\sinh x := \frac{e^{x} - e^{-x}}{2}$$ ← hyperbolic sine

$\cosh(-x) = \cosh x$

$$\cosh x := \frac{e^{x} + e^{-x}}{2}$$ ← hyperbolic cosine



Prop

(i) $$(\sinh x)' = \left(\frac{e^{x} - e^{-x}}{2}\right)'$$

$(-x)'$

$$= \frac{e^{x} - e^{-x}(-1)}{2} = \cosh x$$

(ii) $$(\cosh x)' = \sinh x$$

(iii) $\cosh^2 x - \sinh^2 x = 1$

(iv) $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$

(v) $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$

$$\int u v' \, dx = uv - \int v \cdot u' \, dx$$

<u>Example</u>

$$\int_0^{\ln 2} x \cosh x \, dx$$

$u = x \qquad u' = 1$

$v = \sinh x \qquad v' = \cosh x$

$$= x \sinh x \Big|_0^{\ln 2} - \int_0^{\ln 2} 1 \cdot \underbrace{\sinh x}_{\substack{\parallel \\ (\cosh x)'}} \, dx$$

$$= \ln 2 \cdot \sinh(\ln 2) - 0 \cdot \sinh 0$$

$$- \cosh x \Big|_0^{\ln 2}$$

$$= \ln 2 \sinh(\ln 2) \quad \cosh(\ln 2) + \underset{}{\overset{\overset{1}{\parallel}}{\cosh 0}}$$

$$= \ln 2 \cdot \sinh(\ln 2) - \cosh(\ln 2) + \cosh 0$$

$$\sinh(\ln 2) = \frac{e^{\ln 2} - e^{-\ln 2}}{2} = \frac{2 - \frac{1}{2}}{2} = \frac{3}{4}$$

$$\cosh(\ln 2) = \frac{e^{\ln 2} + e^{-\ln 2}}{2} = \frac{2 + \frac{1}{2}}{2} = \frac{5}{4}$$

$$= \frac{3}{4} \ln 2 - \frac{1}{4} \quad \#$$

Techniques of integration

Recall:

① $\int x^r \, dx = \frac{1}{r+1} x^{r+1} + C$ if $r \neq -1$

② $\int x^{-1} \, dx = \ln|x| + C$

③ $\int e^x \, dx = e^x + C$

④ $\int \sin x \, dx = -\cos x + C$

⑤ $\int \cos x \, dx = \sin x + C$

⑥ $\int \frac{1}{\sqrt{1-x^2}} \, dx = \sin^{-1} x + C$

⑦ $\int \frac{1}{1+x^2} \, dx = \tan^{-1} x + C$

⑧ Integration by subsitution:

$$\int f(u(x)) \cdot u'(x) \, dx = f(u(x)) + C$$

⑨ Integration by parts:

$$\int u(x) \cdot v'(x) \, dx = u(x) \cdot v(x) - \int v(x) \cdot u'(x) \, dx$$

Example

$v' = e^x \quad v = e^x$

$u = x \Rightarrow u' = 1$

① $$\int x e^x \, dx = x e^x - \int 1 \cdot e^x \, dx$$

$$= x e^x - e^x + C \quad \#$$

$$(xe^x - e^x)' = \cancel{1 \cdot e^x} + x \cdot e^x - \cancel{e^x} = x \cdot e^x$$

$v' = x \quad u = \ln x$

② $$\int x \ln x \, dx = \frac{x^2}{2} \cdot \ln x - \int \frac{x^2}{2} \cdot \frac{1}{x} \, dx$$

$v = \frac{x^2}{2} \quad u' = \frac{1}{x}$

$$= \frac{x^2}{2} \ln x - \frac{x^2}{4} + C \quad \#$$

$v' = e^x \quad u = \cos x$

③ $$\int e^x \cos x \, dx$$

$v' = e^x \quad u = \sin x$

$$= e^x \cos x + \int e^x (+\sin x) \, dx$$

$$= e^x \cos x + e^x \sin x - \int e^x \cos x \, dx$$

$$\Rightarrow \quad 2\int e^x \cos x\,dx \text{ "=" } e^x \cos x + e^x \sin x$$

$$\Rightarrow \quad \int e^x \cos x\,dx = \frac{e^x(\cos x + \sin x)}{2} + C \quad \#$$

④ $\int 1 \cdot \sin^{-1} x \, dx = x \cdot \sin^{-1} x - \int x \cdot \frac{1}{\sqrt{1-x^2}}\,dx$

$u = \sin^{-1} x$, $v' = 1$

$-\frac{1}{2}\,du$

$u = 1 - x^2$

$u^{-\frac{1}{2}}$ $\quad du = -2x\,dx$

$$= x \cdot \sin^{-1} x + \int +\frac{1}{2}\,\frac{1}{\sqrt{u}}\,du$$

$$= x \cdot \sin^{-1} x + u^{\frac{1}{2}} + C$$

$$= x \cdot \sin^{-1} x + \sqrt{1-x^2} + C \quad \#$$

Product of trigonometric functions

Goal: Calculate

$$\int \sin^p x \cdot \cos^q x \, dx$$

Recall

(i) $\sin(-\theta) = -\sin\theta$

(ii) $\cos(-\theta) = \cos\theta$

(iii) $\sin^2\theta + \cos^2\theta = 1$

(iv) $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

(v) $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

(vi) $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

(vii) $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

積化和差

(viii) $\sin\alpha\cos\beta = \frac{1}{2}\left(\sin(\alpha+\beta) + \sin(\alpha-\beta)\right)$

$\frac{(iv)+(v)}{2}$ $\frac{(iv)-(v)}{2}$

(ix) $\cos\alpha \sin\beta = \frac{1}{2}\left(\sin(\alpha+\beta) - \sin(\alpha-\beta)\right)$

(x) $\cos\alpha \cos\beta = \frac{1}{2}\left(\cos(\alpha+\beta) + \cos(\alpha-\beta)\right)$

$\frac{(vi)+(vii)}{2}$ $\qquad$ $\frac{(vii)-(vi)}{2}$

(xi) $\sin\alpha \sin\beta = \frac{1}{2}\left(\cos(\alpha-\beta) - \cos(\alpha+\beta)\right)$

$(\alpha=\beta=\theta)$

(xii) $$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

(xiii) $$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

Example

① $\int \sin(5x)\cdot\cos(3x)\,dx$

$$\overset{(viii)}{=} \int \frac{1}{2}\left(\sin(5x+3x) + \sin(5x-3x)\right)dx$$

$\left(\frac{1}{2}\left(-\cos(2x)\right)\right)'$

$= -\frac{1}{2}(-\sin(2x))$

$$= \frac{1}{2}\int \underbrace{\sin(8x)}_{=\left(\frac{1}{8}(-\cos(8x))\right)'} + \sin(2x)\, dx$$

$2\ \cdot (2x)' = \sin(2x)$

$$= \frac{-1}{16}\cos(8x) - \frac{1}{4}\cos(2x)) + C \quad \#$$

② $$\int \sin^2 x \cos^2 x\, dx = \int (\underbrace{\sin x \cdot \cos x}_{= \text{(viii)}})^2 dx$$

$\frac{1}{2}(\sin(x+x) + \sin(x-x)) = \frac{1}{2}\sin(2x)$

$\frac{1-\cos(4x)}{2}$ $=$ (xii)

$$= \frac{1}{4}\int \sin^2(2x)\, dx$$

$$= \frac{1}{8}\int \underbrace{1}_{= x'} - \underbrace{\cos(4x)}_{= \left(\frac{\sin(4x)}{4}\right)'}\, dx$$

$$= \frac{x}{8} - \frac{\sin(4x)}{32} + C \quad \#$$