Calculus 13/12

Main idae: $f(x) = e^{\ln f(x)}$

Example

① $(x^x)' = \left(e^{\ln x^x}\right)'$

$(x>0)$ $= \left(e^{x\ln x}\right)'$

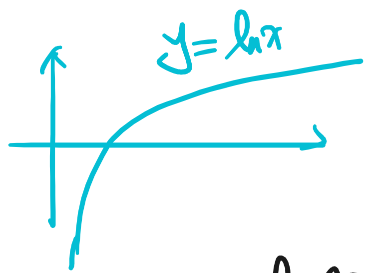
$= e^{x\ln x} \cdot \underline{(x\cdot \ln x)'}$

$\ln x + x\cdot\frac{1}{x}$

$\parallel$

$= x^x \cdot (\ln x + 1)$ #

② $\lim_{x\to 0^+} x^x = ?$



$= \lim_{x\to 0^+} e^{\ln x^x} = \lim_{x\to 0^+} e^{x\ln x}$

since e^x is continuous

$= e^{\lim_{x\to 0^+} x\ln x}$

Recall: if f is continuous at $\lim g(x)$, then $\lim_{x\to c} f(g(x)) = f(\lim_{x\to c} g(x))$

$x \to c$

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x \to -\infty}{\frac{1}{x} \to +\infty}$$

$$= \lim_{x\to 0^+} \frac{(\ln x)' = \frac{1}{x}}{\left(\frac{1}{x}\right)' = -\frac{1}{x^2}} = \lim_{x\to 0^+}(-x)$$

$$= 0$$

$$= e^0 = 1 \quad \#$$

③ $\int_0^1 x\, 5^{-x^2}\, dx$ Let $u = -x^2$

$\Rightarrow du = -2x\,dx$

$-\frac{1}{2}du$

$$= \int_{u(0)=0}^{u(1)=-1} 5^u \left(-\frac{1}{2}\right) du$$

Recall

$$(5^u)' = \left(e^{\ln 5^u}\right)'$$
$$= \left(e^{u\ln 5}\right)'$$
$$= e^{u\ln 5} \cdot \ln 5$$
$$= 5^u \cdot \ln 5$$

$$= -\frac{1}{2}\, \frac{5^u}{\ln 5}\Big|_0^{-1}$$

$$= \ldots \frac{1}{5} \ldots \frac{1}{\ } \frac{1}{\ } = \frac{1}{\ } - \frac{1}{\ }$$

$$= -\frac{}{2}\,\frac{}{\ln 5} + \frac{}{2}\ln 5 - \frac{}{2\ln 5} \quad \frac{}{10\ln 5}$$ #

<u>Remark</u>

Let $a>0$, $a \neq 1$. Then

$$\log_a x = \frac{\log_e x}{\log_e a} = \frac{\ln x}{\ln a}, \quad x>0$$

$$\Rightarrow \quad \frac{d}{dx}(\log_a x) = \left(\frac{\ln x}{\ln a}\right)' = \frac{1}{\ln a}\,\frac{1}{x}$$

<u>Example</u>

① $$\left(\log_5 |x|\right)' = \left(\frac{\ln|x|}{\ln 5}\right)' = \frac{1}{x\cdot \ln 5}$$ #

② $$\int \frac{1}{x\ln 10}\,dx = \frac{\ln|x|}{\ln 10} + C$$

$$= \log_{10}|x| + C$$ #

§ Inverse trigonometric functions

(§7.7)

<u>Def</u>

The restriction of sine function

The restriction of

$$\sin : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \overset{\xleftarrow{\sin^{-1}}}{\longrightarrow} [-1, 1]$$

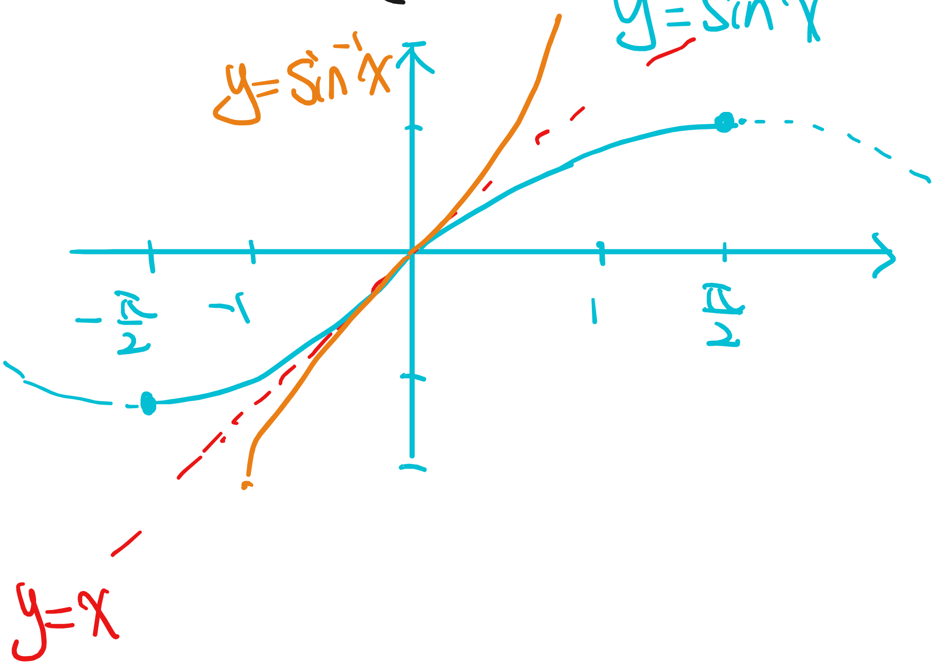
is 1-1 and onto, and its inverse is called arcsine, denoted by

$$\sin^{-1} \quad \text{or} \quad \arcsin$$

In other words,

$$x = \sin^{-1}(\sin x) \qquad \forall x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$y = \sin(\sin^{-1} y) \qquad \forall y \in [-1, 1]$$



e.g.

$$\sin^{-1}(1) = \frac{\pi}{2} = \sin^{-1}\left(\sin\frac{\pi}{2}\right)$$

$$\sin^{-1}(\sin \pi) = 0 \neq \pi$$

$$\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

$$\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$$

Thm

The function $\sin^{-1}$ is differentiable on $(-1, 1)$, and

$$(\sin^{-1}x)' = \frac{1}{\sqrt{1-x^2}}$$

Thus, $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + C$

pf

Let $x = \sin y \Rightarrow \sin^{-1}x = \sin^{-1}(\sin y)$

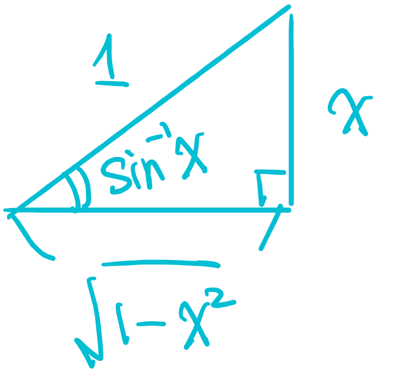
$$= y$$

$$\Rightarrow \frac{d}{dx}(\sin^{-1}x) = \frac{dy}{dx}$$

$$1 = \frac{d}{dx}(x) = \frac{d}{dx}(\sin y)$$

$$= \cos y \cdot \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\cos(\sin^{-1}x)} = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$$



$$\Rightarrow (\sin^{-1}x)' = \frac{1}{\sqrt{1-x^2}} \quad \#$$

Example

① $$\left(\sin^{-1}(3x^2)\right)' = \frac{1}{\sqrt{1-(3x^2)^2}} \cdot (3x^2)'$$

$$= \frac{6x}{\sqrt{1-9x^4}} \quad \#$$

② $$\int_0^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}}\,dx = \int_0^{\sqrt{3}} \underbrace{\frac{1}{2}\frac{1}{\sqrt{1-\left(\frac{x}{2}\right)^2}}\,dx}_{du}$$

$$4-x^2 = 2^2\left(1-\left(\frac{x}{2}\right)^2\right)$$

$u = \frac{x}{2}$

$du = \frac{1}{2}dx$

$$= \int_{u(0)=0}^{u(\sqrt{3})=\frac{\sqrt{3}}{2}} \frac{1}{\sqrt{1-u^2}}\,du$$

$$= \sin^{-1} u \Big|_0^{\frac{\sqrt{3}}{2}}$$

$$= \underbrace{\sin^{-1}\frac{\sqrt{3}}{2}}_{\frac{\pi}{3}} - \underbrace{\sin^{-1} 0}_{=0} = \frac{\pi}{3} \quad \#$$

Def

The restriction of tangent function

$$\tan : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \longrightarrow \mathbb{R}$$

is 1-1 and onto, and its inverse is called arc tangent, denoted by

$$\tan^{-1} x \qquad \text{or} \qquad \arctan x$$

That is

That is,

$$x = \tan^{-1}(\tan x) \qquad \forall x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

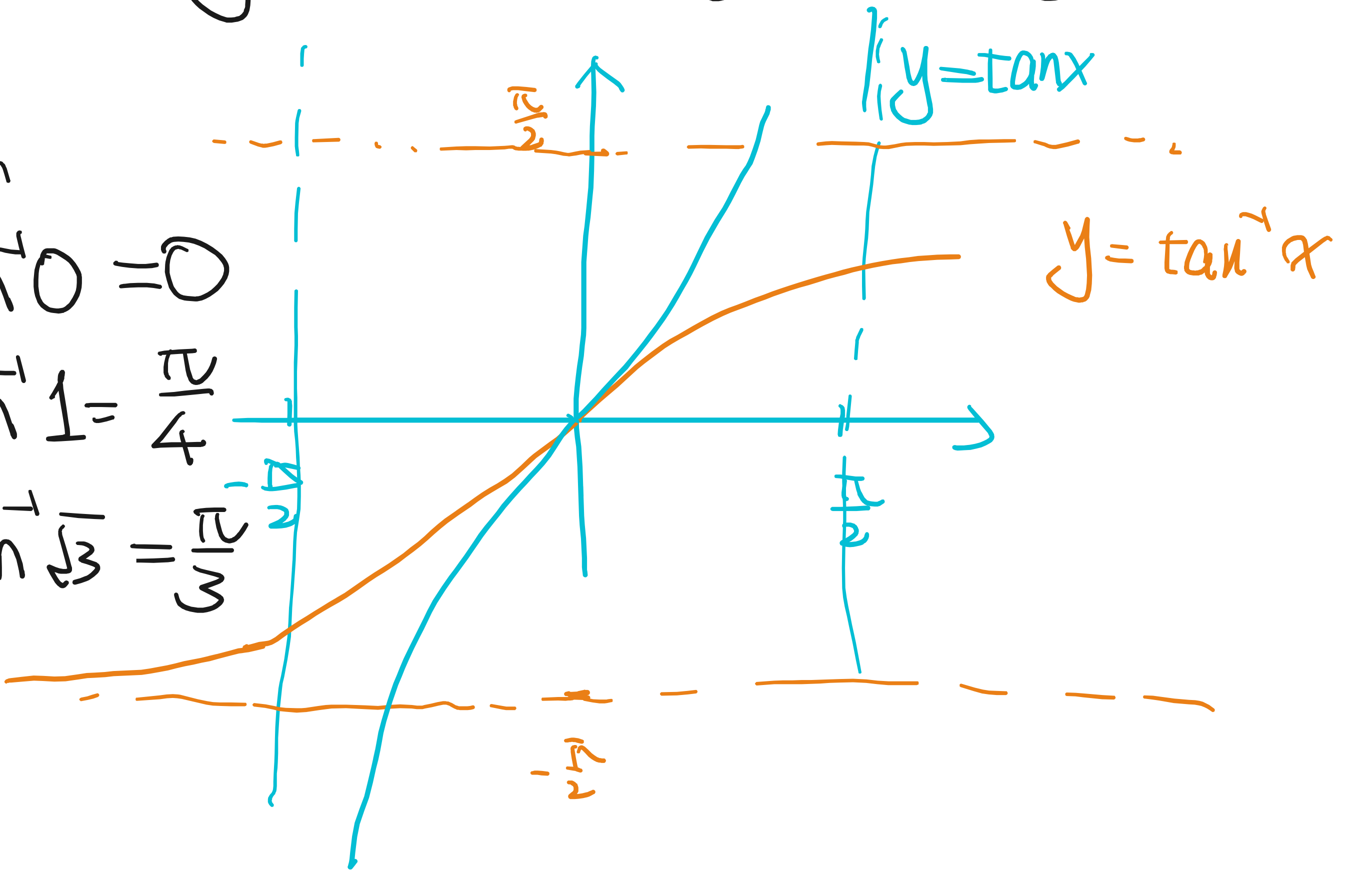
$$y = \tan(\tan^{-1} y) \qquad \forall y \in \mathbb{R}$$

eg.

$\tan^{-1} 0 = 0$

$\tan^{-1} 1 = \frac{\pi}{4}$

$\tan^{-1} \sqrt{3} = \frac{\pi}{3}$



Thm

- $$(\tan^{-1} x)' = \frac{1}{1+x^2}$$

- $$\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C$$

Pf

$$x = \tan(\tan^{-1} x)$$

$$\Rightarrow \quad 1 = \frac{d}{dx}(x) = \frac{d}{dx}\left(\tan(\tan^{-1}x)\right)$$

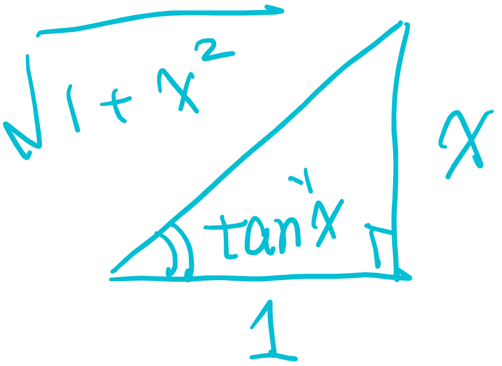
Recall

$(\tan y)' = \sec^2 y$

$$= \sec^2(\tan^{-1}x) \cdot (\tan^{-1}x)'$$

$$\Rightarrow \quad (\tan^{-1}x)' = \frac{1}{\sec^2(\tan^{-1}x)}$$

$$= \cos^2(\tan^{-1}x) = \left(\frac{1}{\sqrt{1+x^2}}\right)^2$$



$$\Rightarrow \cos(\tan^{-1}x) = \frac{1}{\sqrt{1+x^2}}$$

$$= \frac{1}{1+x^2} \quad \#$$

Example

① $$\left(\tan^{-1}\left(\frac{x}{3}\right)\right)' = \frac{1}{1+\left(\frac{x}{3}\right)^2} \cdot \left(\frac{x}{3}\right)'$$

$$= \frac{1}{1+\frac{x^2}{9}} \cdot \frac{1}{3} = \frac{3}{9+x^2} \quad \#$$

② $\displaystyle\int_0^2 \frac{1}{4+x^2}\,dx$

Let
$u = \frac{x}{2}$
$du = \frac{1}{2}dx$

$$= \frac{1}{4}\int_0^2 \frac{1}{1+\left(\frac{x}{2}\right)^2}\,\underline{dx} \quad 2du$$

$$= \frac{1}{2}\int_{u(0)=0}^{u(2)=1} \frac{1}{1+u^2}\,du$$

$$= \frac{1}{2}\tan^{-1}u\,\Big|_0^1 = \frac{1}{2}\Big(\underbrace{\tan^{-1}1}_{=\frac{\pi}{4}} - \underbrace{\tan^{-1}0}_{=0}\Big)$$

$$= \frac{\pi}{8} \quad \#$$

HW9. 8

$f(x) = \int_2^x \sqrt{1+t^2}\, dt$

(a) f is 1-1

(b) $(f^{-1})'(0) = ?$

Sol

(a) $f'(x) = \sqrt{1+x^2} \geq \sqrt{1} > 0$

$\Rightarrow$ f is strictly increasing

$\Rightarrow$ f is 1-1 because

$$x \neq y \Rightarrow \left(\begin{array}{l} x < y \Rightarrow f(x) \lneqq f(y) \\ \\ x > y \Rightarrow f(x) \gneqq f(y) \end{array} \right)$$

$\Rightarrow$ $f(x) \neq f(y)$ #

(b) Note that

$$f(2) = \int_2^2 \sqrt{1+t^2}\, dt = 0$$

$\Rightarrow$ $f^{-1}(0) = 2$

$$(f^{-1})'(0) = \frac{1}{f'(2)}$$

$$= \frac{1}{\sqrt{1+2^2}} = \frac{1}{\sqrt{5}} \quad \#$$

HW9.5 4

Prove

$$\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = e$$

pf

$$e^{\ln\left(1+\frac{1}{x}\right)^x} = e^{x\ln\left(1+\frac{1}{x}\right)}$$

1 ?

$$\ln\left(1+\frac{1}{x}\right) = \int_1^{1+\frac{1}{x}} \frac{1}{t}\,dt$$

$1 \le t \le 1+\frac{1}{x}$

$$\frac{x}{1+x} = \frac{1}{1+\frac{1}{x}} \le \frac{1}{t} \le \frac{1}{1} = 1$$

$$\ln\left(1+\frac{1}{x}\right) = \int_1^{1+\frac{1}{x}} \frac{1}{t}\,dt$$

$$\int_1^{1+\frac{1}{x}} \frac{x}{1+x}\,dt \le \int_1^{1+\frac{1}{x}} 1\,dt$$

$$\frac{x}{1+x}\cdot\left(1+\frac{1}{x}-1\right) = \frac{1}{1+x} \qquad 1\cdot\left(1+\frac{1}{x}-1\right) = \frac{1}{x}$$

$(x>0)$

$\Rightarrow$

$$\frac{1}{x+1}\cdot x \le x\cdot\ln\left(1+\frac{1}{x}\right) \le x\cdot\frac{1}{x} = 1$$

$$\frac{x}{x+1} \to 1 \text{ as } x\to\infty \qquad 1 \to 1$$

By Pinching Thm,

$$\lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right) = 1$$

$\Rightarrow$

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty} e^{x\ln\left(1+\frac{1}{x}\right)}$$

$$= e^{\lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right)} = e^1 = e$$

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