Calculus 12/7

Recall

$$\ln x = \int_1^x \frac{1}{t}\,dt \quad , \quad x>0$$

The function

$$\ln : (0,\infty) \longrightarrow (-\infty,\infty)$$

is 1-1, onto, and thus invertible

Def

The (natural) exponential function

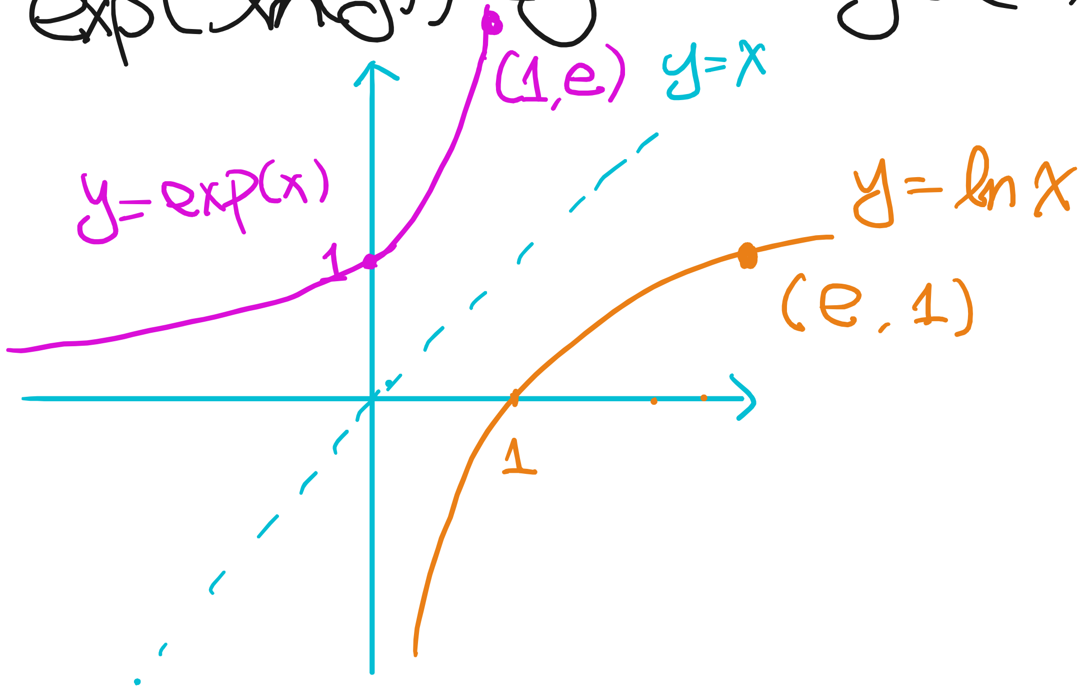
$$\exp : (-\infty,\infty) \to (0,\infty)$$

is defined to be the inverse of $\ln : (0,\infty) \longrightarrow (-\infty,\infty)$.

That is,

$$\ln(\exp(x)) = x \qquad \forall x \in (-\infty,\infty)$$

$$\exp(\ln(y)) = y \quad \forall y \in (0,\infty)$$



Thm (§7.4)

(i) $\exp(0) = 1, \quad \exp(1) = e$

(ii) $\exp(x) > 0 \quad \forall x \in \mathbb{R}$

(iii) $\lim_{x \to -\infty} \exp(x) = 0$

(iv) $\exp$ is strictly increasing.

(v) $\exp(a+b) = \exp(a) \cdot \exp(b) \quad \forall a,b \in \mathbb{R}$

pf

Recall: $\ln(x \cdot y) = \ln x + \ln y \quad \forall x,y > 0$

Let $x = \exp(a)$, $y = \exp(b)$

$\Rightarrow \ln(\exp(a)\cdot\exp(b))$

$= \ln(\exp(a)) + \ln(\exp(b))$

$= a+b$

$\Rightarrow \boxed{\exp(a)\cdot\exp(b) = \exp(a+b)} = \exp(\ln(\exp(a)\cdot\exp(b)))$

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(vi) $\exp(n) = \exp(\overbrace{1+1+\cdots+1}^{n \text{ times}})$

$\overset{(v)}{=} \underbrace{\exp(1)}_{\overset{||(i)}{e}} \cdot \exp(n-1) = \cdots = e^n$

(vii) $\exp(\frac{1}{n}) = e^{\frac{1}{n}}$

pf

$\overbrace{\underbrace{\exp(\frac{1}{n})\cdot\exp(\frac{1}{n})}_{||(v)} \cdot \cdots \cdot \exp(\frac{1}{n})}^{n \text{ times}}$

$\exp(\frac{1}{n}+\frac{1}{n}) = \exp(\frac{2}{n})$

$= \exp(\overbrace{\frac{1}{n}+\cdots+\frac{1}{n}}^{n \text{ times}}) = \exp(1) = e$

$\Rightarrow \exp(\frac{1}{n}) = \sqrt[n]{e} = e^{\frac{1}{n}}$

#

(viii)

$$\underline{\exp\left(\frac{p}{q}\right)} = \exp\Big(\overbrace{\tfrac{1}{q}+\cdots+\tfrac{1}{q}}^{p\text{ times}}\Big)$$

$$\overset{(v)}{=} \left(\exp\left(\frac{1}{q}\right)\right)^{p} \overset{(vii)}{=} \left(e^{\frac{1}{q}}\right)^{p} = \underline{e^{\frac{p}{q}}}$$

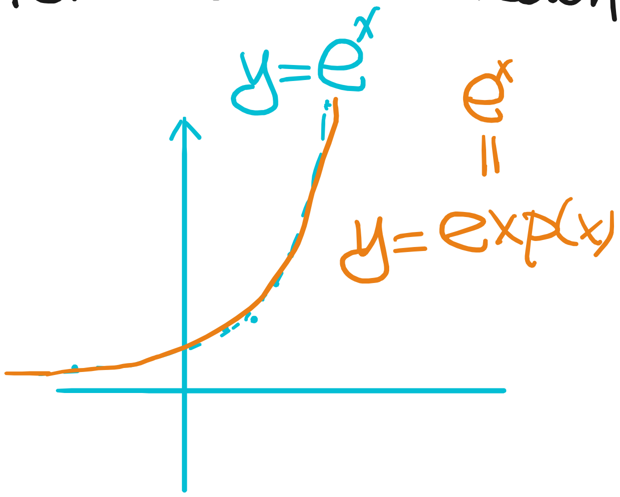
<u>Remark</u>

① (irrational power)

$\exp(x)$ is the continuous function

s.t.

$$\exp\left(\frac{p}{q}\right) = e^{\frac{p}{q}}$$



We define

$$e^x = \exp(x)$$

for irrational numbers x.

② Since $\ln$ is the inverse of exp,

$$\ln x = \log_e x$$

<u>Thm</u> (Thm 7.4.9)

The function $\exp(x) = e^x$ is differentiable the function

at any $x \in \mathbb{R}$, and

$$\boxed{\frac{d}{dx}(e^x) = e^x}$$

Thus,

$$\int e^x \, dx = e^x + C.$$

<u>pf</u>

Since

$$x = \ln(e^x),$$

we have

$$1 = \frac{d}{dx}(x) = \frac{d}{dx}\left(\ln(e^x)\right)$$

$$\underset{\text{chain rule}}{=} \frac{1}{e^x} \cdot \frac{d}{dx}(e^x)$$

$$\Rightarrow \frac{d}{dx}(e^x) = e^x \qquad \#$$

<u>Example</u>

① $\frac{d}{dx}(e^{3x}) \underset{\text{rule}}{\overset{\text{Chain}}{=}} e^{3x} \cdot (3x)' = 3e^{3x}$

② $\frac{d}{dx}(e^{\sqrt{x}}) = e^{\sqrt{x}} \cdot (\sqrt{x})' = e^{\sqrt{x}} \frac{1}{2\sqrt{x}}$

③ $\frac{d}{dx}(xe^{-x}) = (x)' \cdot e^{-x} + x \cdot (e^{-x})'$

$= e^{-x} - xe^{-x}$

④ $\int_0^1 \underbrace{9e^{3x}}_{3\cdot(e^{3x})'} dx = 3e^{3x}\Big|_0^1 = 3e^3 - 3$

⑤ $\int_0^1 \frac{e^{3x}\,dx}{e^{3x}+1}$ $\quad e^{3x}dx = \frac{1}{3}du$

Let $u = e^{3x}$

$\Rightarrow du = 3e^{3x}dx$

$= \int_{u(0)=1}^{u(1)=e^3} \frac{1}{u+1} \frac{1}{3} du$

$= \frac{1}{3}\ln|u+1|\Big|_1^{e^3} = \frac{1}{3}\ln(e^3+1) - \frac{1}{3}\ln 2$

$$= \frac{1}{3}\ln\left(\frac{e^3+1}{2}\right) \quad \#$$

Recall:

$$\int u'v\,dx = uv - \int u\cdot v'\,dx$$

(6) $\int_0^1 x e^x\,dx$

$$= xe^x\Big|_0^1 - \int_0^1 1\cdot e^x\,dx$$

Let

$u' = e^x \quad u = e^x$

$v' = x \quad v = x$

$$= 1\cdot e - 0\cdot e^0 - e^x\Big|_0^1$$

$$= e - (e - e^0) = 1 \quad \#$$

Arbitrary power

Q: $10^\pi = ?$

Def

Let $a > 0$. Define (for $x \notin \mathbb{Q}$) 有理數

$$a^x := \exp(x\cdot\ln a)$$

$$= e^{x \cdot \ln a}$$

idea:

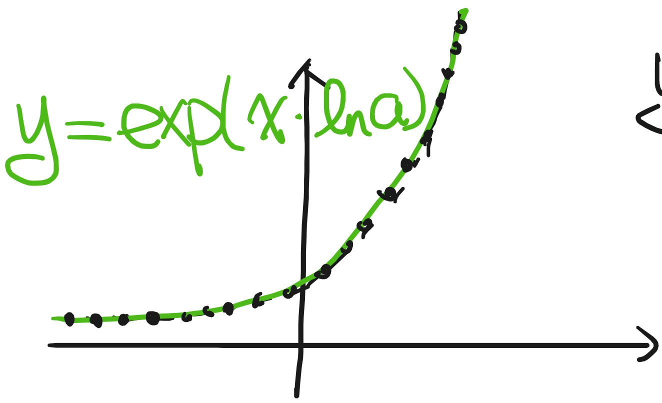
$\ln a^x \overset{\text{hope}}{=} x \ln a$

$a^x = e^{\ln a^x}$

$= e^{x \ln a}$

continuous

$= a^{\frac{p}{q}}$ if $x = \frac{p}{q} \in \mathbb{Q}$



$y = a^x, \ x \in \mathbb{Q}$

<u>Thm</u> (7.5.1)

For any $x, y \in \mathbb{R}$, $a > 0$,

(i) $a^{x+y} = a^x \cdot a^y$

(ii) $a^{x-y} = \dfrac{a^x}{a^y}$

(iii) $(a^x)^y = a^{xy}$

<u>pf of (i)</u> ((ii), (iii) are similar).

$$a^{x+y} = \exp(\underbrace{(x+y) \cdot \ln a}_{x \cdot \ln a + y \cdot \ln a})$$

$$= \exp(x \cdot \ln a) \cdot \exp(y \cdot \ln a)$$

$$= a^x \cdot a^y \qquad \#$$

<u>Thm</u> (7.5.3, 7.5.4)

Let $r \in \mathbb{R}$, $x > 0$.

$$(x^r)' = r \cdot x^{r-1}$$

$$\int x^r dx = \frac{x^{r+1}}{r+1} + C.$$

<u>pf</u>

$$(x^r)' = \left(\exp(r \cdot \ln x)\right)'$$

$$\overset{\text{chain rule}}{=} \underbrace{\exp(r \cdot \ln x)}_{x^r} \cdot \underbrace{(r \cdot \ln x)'}_{\frac{r}{x}}$$

$$= r\, x^{r-1} \qquad \#$$

<u>Thm</u>

Let $a > 0$

Let $a > 0$.

$$(a^x)' = a^x \cdot \ln a$$

$$\int a^x \, dx = \frac{1}{\ln a} a^x + C$$

pf

$$(a^x)' = \big(\exp(x \cdot \ln a)\big)'$$

$$= \overset{a^x}{\exp(x \cdot \ln a)} \cdot \overset{\overset{\ln a}{\shortparallel}}{(x \cdot \ln a)'}$$

$$= a^x \cdot \ln a \qquad \#$$

Example

① $$\frac{d}{dx}\Big(3^{\overset{\overset{y}{\shortparallel}}{x^2}}\Big) \overset{\text{chain}}{\underset{\text{rule}}{=}} \frac{d3^y}{dy} \cdot \frac{dy}{dx}$$

$$= 3^y \cdot \ln 3 \cdot 2x$$

$$= 3^{x^2} \cdot 2\ln 3 \cdot x \qquad \#$$

② $\big((x^2 \ldots)^{3x}\big)' \qquad \big(e^{\ln(x^2+1)^{3x}}\big)'$

② $\left((x^2+1)^{3x}\right)' = \left(e^{3x\cdot\ln(x^2+1)}\right)'$

$$= \left(e^{3x\cdot\ln(x^2+1)}\right)'$$

chain rule

$$= e^{3x\cdot\ln(x^2+1)} \cdot \boxed{\left(3x\cdot\ln(x^2+1)\right)'}$$

$(x^2+1)^{3x}$ //

// product rule

$$(3x)'\cdot\ln(x^2+1) + 3x\cdot\left(\ln(x^2+1)\right)'$$

$$= 3\cdot\ln(x^2+1) + 3x\cdot\frac{1}{x^2+1}\cdot(x^2+1)'$$

$$= 3\ln(x^2+1) + 6\frac{x^2}{x^2+1}$$

$$= (x^2+1)^{3x}\cdot\left(3\ln(x^2+1) + \frac{6x^2}{x^2+1}\right) \quad \#$$