Calculus 12/5

Thm (Thm 7.18)

Let f be a 1-1, differentiable function, and let $f(a) = b$.

If $f'(a) \neq 0$, then f^{-1} is differentiable at b, and

$$\boxed{(f^{-1})'(b) = \frac{1}{f'(a)}}$$

pf (Skip the differentiability of f^{-1})

Since

$$x = f^{-1}(f(x)),$$

by applying $\frac{d}{dx}$ to both sides,

c.f. implicit differentiation

we have

$$\frac{d}{dx}(x) = \underset{\sim}{1} = \frac{d}{dx}\left(f^{-1}(f(x))\right)$$

chain rule

$$= \underbrace{(f^{-1})'(f(x)) \cdot f'(x)}_{\sim}$$

$$\Rightarrow 1 = (f^{-1})'(\boxed{f(a)}) \cdot \underset{\sim}{f'(a)}$$

$b = f(a)$, $f'(a) \neq 0$

$$\Rightarrow (f^{-1})'(b) = \frac{1}{f'(a)} \quad \#$$

Example

Let

$$f(x) = x^3 + \frac{1}{2}x$$

Find $(f^{-1})'(9)$.

$9 = b = f(a)$

Sol

1° Since $f'(x) = 3x^2 + \frac{1}{2} \geq \frac{1}{2} > 0,$

f is 1-1, differentiable.

2° Solve

$$f(a) = 9 = a^3 + \frac{1}{2}a$$

$\Rightarrow \quad a = 2$

3° So

$$(f^{-1})'(9) = \frac{1}{f'(a)} = \frac{1}{3 \cdot 2^2 + \frac{1}{2}}$$

$$= \frac{2}{25} \qquad \#$$

Or use:

$$x = f^{-1}(f(x)) = f^{-1}(x^3 + \frac{1}{2}x)$$

$$\Rightarrow \quad 1 = \frac{d}{dx}\left(f^{-1}(x^3 + \frac{1}{2}x)\right)$$

$$= \left((f^{-1})'(x^3 + \frac{1}{2}x)\right) \cdot \left(3x^2 + \frac{1}{2}\right)$$

$$\overset{x=2}{\Rightarrow} \quad 1 = (f^{-1})'(9) \cdot \frac{25}{2}$$

$$\Rightarrow (f^{-1})'(9) = \frac{2}{25} \quad \#$$

Logarithm function

Recall:

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$$

for $n \neq -1$

Question:

$$\int x^{-1} \, dx = ?$$

That is, is there a function f s.t.

$$f'(x) = \frac{1}{x} \; ?$$

Def (§7.2)

The function $\log_e x$ 定義成

$$\ln x \; (= \log x) := \int_1 \frac{1}{t}\,dx$$

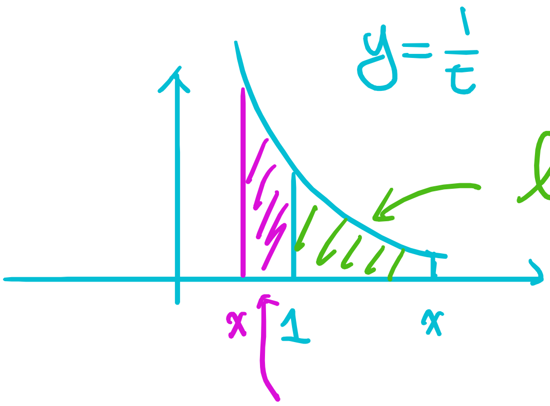
, for $x > 0$, is called

the (natural) logarithm function

Thm (§7.3) ∵ $\frac{1}{t}$ is positive and continuous on $(0,\infty)$ and foundamental thm of Calculus

(i) The function $\ln x$ is differentiable function

$$\ln : (0,\infty) \to (-\infty,\infty)$$



$\ln x = \int_1^x \frac{1}{t}\,dt$

$-\ln x$

(ii) $\frac{d}{dx}(\ln x) = \frac{1}{x} > 0$ for $x > 0$

And

$$\frac{d}{dx}(\ln|x|) = \frac{1}{x} \quad \forall x \neq 0$$

$\frac{d}{dx}(\ln|x|) = \frac{1}{x} \quad \forall x \neq 0$

If $x < 0$,

$$\ln|x| = \ln(-x)$$

$$\Rightarrow \frac{d}{dx}(\ln|x|) = \frac{d}{dx}(\ln(-x))$$

$$\overset{\text{chain}}{\underset{\text{rule}}{=}} \frac{1}{\underbrace{(-x)}_{\ln'(-x)}} \cdot \underbrace{(-x)'}_{-1} = \frac{1}{x} \quad \#$$

(iii) $\ln : (0,\infty) \to (-\infty, \infty)$ is strictly increasing.

(iv)
$$\ln x \begin{cases} > 0, & x > 1, \\ = 0, & x = 1, \\ < 0, & x < 1 \end{cases}$$

<u>Thm</u> (§7.2)

For $a, b > 0$, $\forall$ integers p, q

(i) $\ln(ab) = \ln a + \ln b$

$\ln(ab) = \ln a + \ln b$

(ii) $\ln\left(\frac{1}{b}\right) = -\ln b$

(iii) $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$

(iv) $\ln\left(a^{\frac{p}{q}}\right) = \frac{p}{q} \cdot \ln a.$

pf

Consider

$$\frac{d}{dx}\left(\ln(x \cdot b)\right) \overset{\text{chain}}{\underset{\text{rule}}{=}} \frac{1}{x \cdot b} \cdot \underbrace{(x \cdot b)'}_{=b}$$

$$= \frac{1}{x} = \frac{d}{dx}(\ln x) \quad \forall x > 0.$$

Recall

$$f'(x) = g'(x) \quad \forall x \in (a,b) \Rightarrow f(x) = g(x) + C \text{ on } (a,b)$$

for some constant C

So

$$\ln(xb) = \ln x + C \quad \forall x > 0$$

$$\ln(x \cdot b) = \ln x + C, \quad \forall x > 0,$$

for some constant C.

Taking $x = 1$,

$$\ln(1 \cdot b) = \ln 1 + C = C$$

$$\Rightarrow \quad C = \ln b$$

$$\Rightarrow \quad \ln(x \cdot b) = \ln x + \ln b$$

(i)

$$\Rightarrow \quad \boxed{\ln(a \cdot b) = \ln a + \ln b} \quad \#$$

(ii) $\quad 1 = b \cdot \frac{1}{b}$

$$\Rightarrow \quad \underbrace{\ln(1)}_{0} = \ln\left(b \cdot \frac{1}{b}\right) \overset{(i)}{=} \ln b + \ln\left(\frac{1}{b}\right)$$

$$\Rightarrow \quad \ln\left(\frac{1}{b}\right) = -\ln b \quad \#$$

(iii) $\quad \ln\left(\frac{a}{b}\right) = \ln\left(a \cdot \frac{1}{b}\right) \overset{(i)}{=} \ln(a) + \ln\left(\frac{1}{b}\right)$

$$\overset{(ii)}{=} \ln a - \ln b. \quad \#$$

(iv) $\ln(a^{\frac{p}{q}}) = \ln((a^{\frac{1}{q}})\cdot(a^{\frac{1}{q}})\cdots(a^{\frac{1}{q}}))$

$$\overset{(i)}{=} \underbrace{\ln(a^{\frac{1}{q}}) + \ln(a^{\frac{1}{q}}) + \cdots + \ln(a^{\frac{1}{q}})}_{p}$$

$$= p\cdot \ln(a^{\frac{1}{q}})$$

Moreover, since

$$(a^{\frac{1}{q}})^{q} = a,$$

we have

$$\ln((a^{\frac{1}{q}})^{q}) = \ln a$$

$$= q\cdot \ln(a^{\frac{1}{q}})$$

$$\Rightarrow \ln(a^{\frac{1}{q}}) = \frac{1}{q}\ln a$$

So $\ln(a^{\frac{p}{q}}) = p\ln(a^{\frac{1}{q}}) = \frac{p}{q}\ln a$

#

Remark

Sem

Since
$$\ln a^{\frac{p}{q}} = \frac{p}{q} \ln a,$$
it can be as large as you want.
and as small "

That is,
$$\ln : (0,\infty) \longrightarrow (-\infty, \infty)$$
is onto (also we know it's 1-1)

So it is invertible!!

So there exists a unique number
$$e \in (0, \infty)$$
s.t.
$$\ln e = 1$$

Def (§7.2)

The number e is the unique

number satisfying

$$\ln(e) = 1 = \int_1^e \frac{1}{t}\, dt$$

Remark

That is, it does NOT satisfy algebraic equation

$P(x) = 0$

polynomial

$$e = 2.718 \ldots\ldots$$

is a "transcendental number"

and, in particular, an irrational number

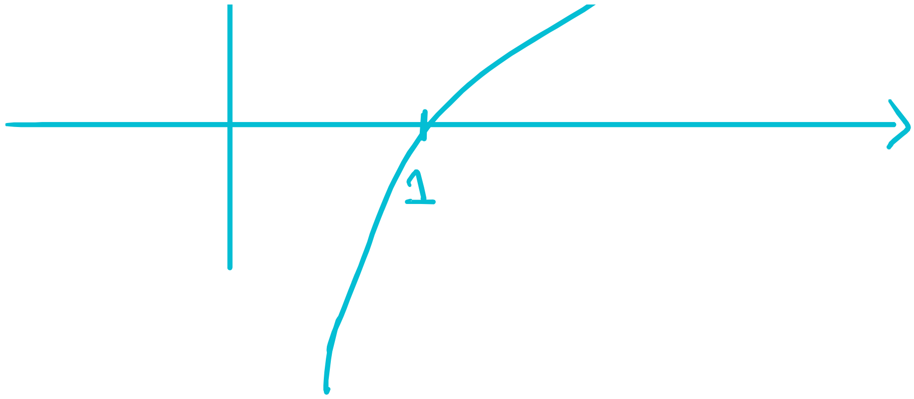
$P(e) \neq 0$

$\forall$ polynomial $P(x)$

Remark

x	1	2	e	3	4
$\ln x$	0	0.69	1	1.1	1.39

approximation 近似值



Remark

Since

$$\frac{d}{dx}\left(\ln|x|\right) = \frac{1}{x} \quad \forall x \neq 0.$$

$$\boxed{\int \frac{1}{x}\,dx = \ln|x| + C}$$

Example

① $\displaystyle\int_1^2 \frac{6x^2+2}{x^3+x+1}\,dx = ?$ $\quad (6x^2+2)\,dx = 2du$

sol

Let $u = x^3+x+1 \Rightarrow du = (3x^2+1)\,dx$

$$\int_1^2 \frac{6x^2+2}{x^3+x+1}\,dx = \int^{u(2)=11} \frac{1}{u}\,2du$$

$$u(1) = 3$$

$$= 2\ln|u| \Big|_{u=3}^{11}$$

$$= 2(\ln 11 - \ln 3) = 2\ln\frac{11}{3} \quad \#$$

$$= -(\cos x)'$$

② $\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx$

(Let $u = \cos x \Rightarrow du = -\sin x \cdot dx$)

$$= \int \frac{1}{u} (-1) \, du$$

$$= -\ln|u| + C$$

$$= -\ln|\cos x| + C$$

$$= \ln\left|\frac{1}{\cos x}\right| + C = \ln|\sec x| + C$$

$$\int \tan x \, dx \;\; || \;\; \ln|\sec x| + C$$

\#

Similarly

$$\int \cot x \, dx = \ln|\sin x| + C$$

$$\int \sec x \, dx = \ln|\sec x + \tan x| + C$$

$$\int \csc x \, dx = \ln|\csc x - \cot x| + C$$