Calculus 11/30

Inverse function

sets 集合

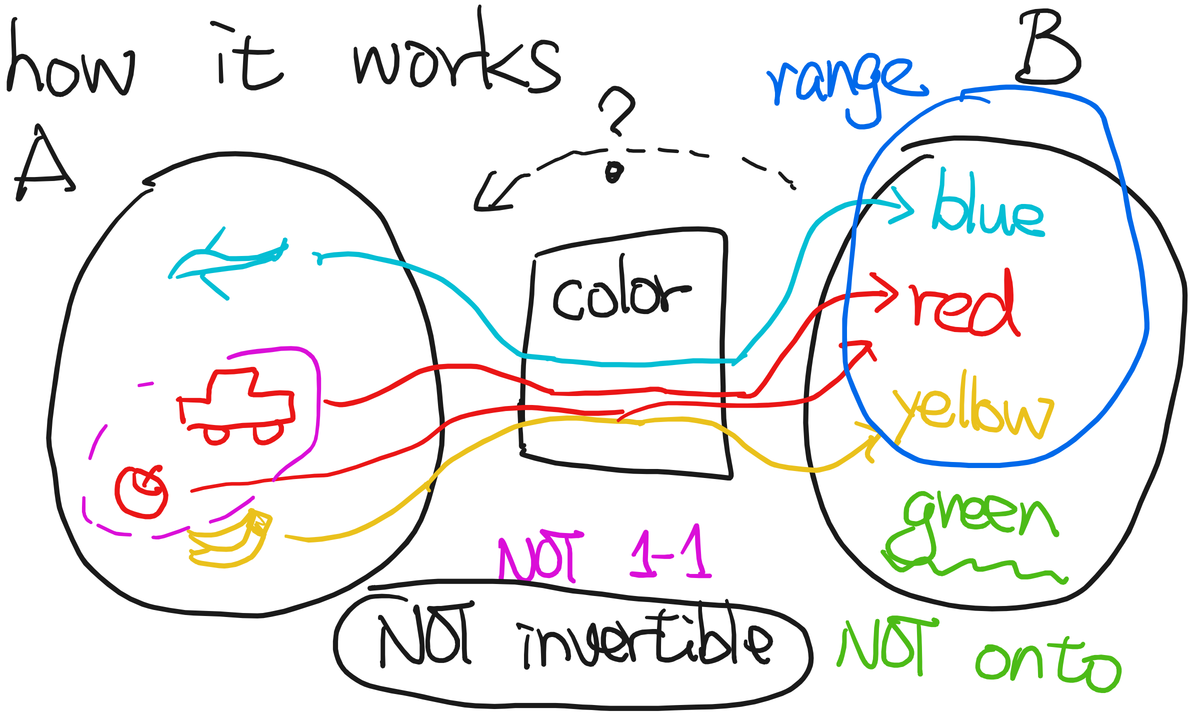
Recall that a function $f: A \to B$ has 3 ingredients:

(i) domain = A = {possible inputs}
定義域

(ii) Codomain = B = {possible outputs}
對應域

(iii) how it works

eg.



Def

Let $f: A \to B$ be a function.

— 對 —

(i) f is one-to-one (or 1-1)
if f satisfies the condition:
if $x_1, x_2 \in A$, $x_1 \neq x_2$, then
$f(x_1) \neq f(x_2)$

(ii) The $\overset{\text{值域}}{\text{range}}$ (or image) of f
is the set ← 集合 = 一堆東西　满足
$$\text{range}(f) = \text{im}(f) = \{\overbrace{f(x)} \in B \mid x \in A\}$$

(iii) f is onto if $\text{im}(f) = B$

Example

① $f : \overset{\overset{\text{domain}}{\|}}{(0,\infty)} \longrightarrow \overbrace{(0,\infty)}^{\text{codomain}}$,

$f(x) = \dfrac{1}{x}$ ← how it works

f is 1-1 and onto

$x_1 \neq x_2 \Rightarrow \dfrac{1}{x_1} \neq \dfrac{1}{x_2}$

codomain
$\|$
$\forall y \in (0,\infty)$,
$\exists x = \dfrac{1}{y} \in \overset{\overset{\text{domain}}{\|}}{(0,\infty)}$
s.t. $f\left(\dfrac{1}{y}\right) = y$

↓

② $g : (-\infty, 0) \cup (0, \infty) \longrightarrow (-\infty, \infty)$

$g(x) = \frac{1}{x}$

g is 1-1, NOT onto

Consider $0 \in (-\infty, \infty)$

$g(x) = \frac{1}{x} \neq 0$

$\forall x \in (-\infty, 0) \cup (0, \infty)$

$\Rightarrow 0 \notin \mathrm{im}(g) \Rightarrow \mathrm{im}(g) \neq (-\infty, \infty)$

Def

A function $f: A \to B$ is invertible 可逆

if $\exists\ g: B \to A$ s.t.

$$g(f(x)) = x \quad \forall x \in A$$

$$f(g(y)) = y \quad \forall y \in B$$

Such a function $g: B \to A$ is called the inverse 反函數 of f, denoted by $g = f'$

Thm

A function $f: A \to B$ is invertible if and only if f is $\underset{\sim}{1\text{-}1}$ and $\underset{\sim}{\text{onto}}$.

↑ more important

Remark

If $f: A \to B$ is 1-1, then its "corestriction"

$$f: A \longrightarrow \operatorname{im}(f)$$

is 1-1 and onto, and thus invertible. (See Thm 7.1.2)

Example

① $f(x) = x^3$

$$f: (-\infty, \infty) \to (-\infty, \infty)$$

is $\underset{\sim}{1\text{-}1}$ and $\underset{\sim}{\text{onto}}$ $\Rightarrow$ invertible

$x_1 \neq x_2 \Rightarrow x_1^3 \neq x_2^3$

$\forall y \in (-\infty, \infty)$, $\exists\, x = \sqrt[3]{y}$ s.t.

$f(\sqrt[3]{y}) = (\sqrt[3]{y})^3 = y$

$f(\sqrt[3]{y}) = (\sqrt[3]{y})^3 = y$

Its inverse $f^{-1}: (-\infty, \infty) \to (-\infty, \infty)$ satisfies the equation

$$f(f^{-1}(y)) = y$$

$$\overset{\|}{(f^{-1}(y))^3} = y \leftarrow$$

Solve x in $f(x) = y$

$$\Rightarrow \quad f^{-1}(y) = \sqrt[3]{y} \qquad \#$$

② $f: (-\infty, \infty) \to (-\infty, \infty)$

$f(x) = 3x - 5$ is 1-1, onto

$f^{-1}(y) = ?$

$$y = f(f^{-1}(y)) = \boxed{3\,f^{-1}(y) - 5 \quad = y}$$

$$\Rightarrow \quad f^{-1}(y) = \frac{y+5}{3} \qquad \#$$

③ $f: (-\infty, \infty) \longrightarrow (-\infty, \infty)$

$x^2 \neq -1 \quad \forall x \in (-\infty, \infty)$

$f(x) = x^2$ is NOT 1-1, NOT onto

$-1 \neq 1$, but $(-1)^2 = 1^2$

$\Rightarrow \quad f: (-\infty, \infty) \to (-\infty, \infty)$ is NOT invertible.

$\overset{\shortparallel}{\mathbb{R}}$ #

④ Show $f: \mathbb{R} \longrightarrow \mathbb{R}$,

$$f(x) = x^5 + 2x^3 + 3x - 4$$

is 1-1

<u>pf</u>

Note that

$$f'(x) = 4x^4 + 2x^2 + 3 \geqslant 3 > 0$$

$\Rightarrow$ f is strictly increasing, i.e.

$$x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$$

So

$$x_1 \neq x_2 \Rightarrow \begin{cases} x_1 < x_2 \Rightarrow f(x_1) < f(x_2) \\ \qquad \Rightarrow f(x_1) \neq f(x_2) \\ x_2 < x_1 \Rightarrow f(x_2) < f(x_1) \\ \qquad \Rightarrow f(x_1) \neq f(x_2) \end{cases}$$

$$\Rightarrow f(x_1) \neq f(x_2)$$

So f is 1-1. #

National Tsing Hua University

Calculus I – Exam 2

Instructor: Hsuan-Yi Liao

Fall, 2023

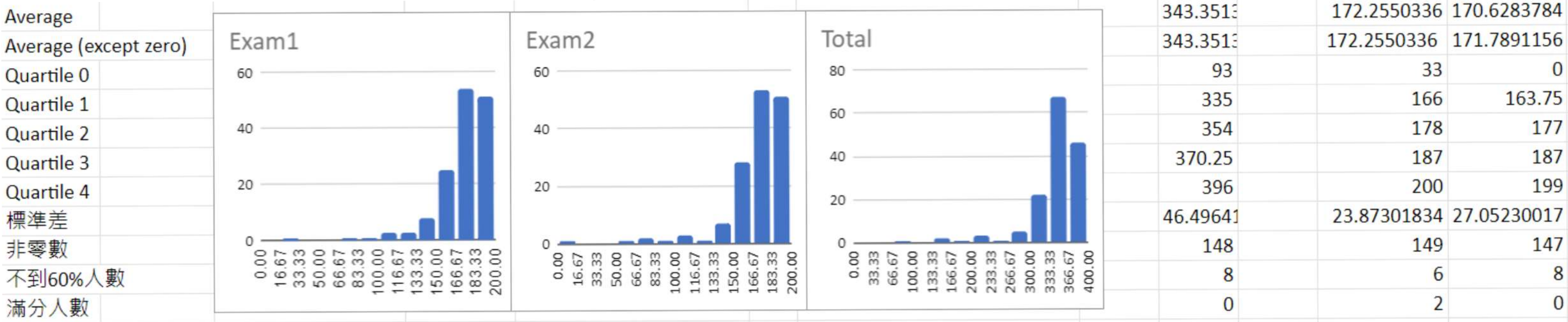
Name: Answer

Student ID: ______________________

- This exam contains 10 pages (including this cover page) and 11 questions.
- Total of points is 200.
- Time limit: **100 minutes**.
- Write down your computation or arguments in details unless otherwise stated.
- The use of a calculator, cell phone, or any other electronic device is **NOT** permitted.
- The use of books or notes of any kind is **NOT** permitted.
- The use of L'Hôspital's rule is **NOT** allowed in this exam.

Distribution of Marks

Question	Points	Score
1	30	
2	10	
3	15	
4	40	
5	15	
6	15	



Average	343.3513	172.2550336	170.6283784
Average (except zero)	343.3513	172.2550336	171.7891156
Quartile 0	93	33	0
Quartile 1	335	166	163.75
Quartile 2	354	178	177
Quartile 3	370.25	187	187
Quartile 4	396	200	199
標準差	46.49641	23.87301834	27.05230017
非零數	148	149	147
不到60%人數	8	6	8
滿分人數	0	2	0

1. Decide whether or not the indicated limit exists. Evaluate the limits that do exist. If the limit does not exist, explain why.

   (a) (6 points) $\lim_{x\to 2} f(x)$ if $f(x) = \begin{cases} 3, & x \text{ an integer} \\ x+2, & \text{otherwise.} \end{cases}$ $= 4$

   (b) (8 points) $\lim_{x\to 0^+} \frac{\sin x}{\sqrt{x}}$. $= \lim_{x\to 0^+} \overset{0}{\sqrt{x}} \overset{\to 1}{\boxed{\frac{\sin x}{x}}} = 0$

   (c) (8 points) $\lim_{x\to 0} \frac{\sqrt{2+x}-\sqrt{2-x}}{x}$. $= \lim_{x\to 0} \frac{\frac{1}{2}(2+x)^{-\frac{1}{2}} - \frac{1}{2}(2-x)^{-\frac{1}{2}}(-1)}{1} = \frac{1}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} = \frac{1}{\sqrt{2}}$

   (d) (8 points) $\lim_{x\to(\pi/2)^-} (x-\frac{\pi}{2})\sec x$. $= \lim_{x\to\frac{\pi}{2}^-} \frac{x-\frac{\pi}{2}}{\cos x} = \lim_{x\to\frac{\pi}{2}^-} \frac{1}{-\sin x}$

$= -1$

2. (10 points) Evaluate the limit $\lim_{x\to 0} \sin(x^2)\cos(\frac{1}{x})$. Prove your answer.

$0 \leq |\sin(x^2)\cos(\frac{1}{x})| \leq |\sin(x^2)| \to 0$ as $x \to 0$

Pinching Thm $\Rightarrow$ $\lim_{x\to 0} \sin(x^2)\cos(\frac{1}{x}) = 0$ #

(see Exam 1)

3. (15 points) Let

$$f(x) = \begin{cases} x^2 \sin(\frac{1}{x}), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

$\lim_{x \to 0} f'(x)$ and 可微 are Different !!

Is f differentiable at $x = 0$? Is f twice differentiable (i.e. f' is differentiable) at $x = 0$? Explain your answers.

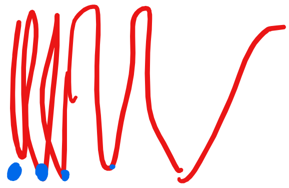
(i) $$\left| \frac{h^2 \sin(\frac{1}{h}) - f(0)}{h} \right| = \left| h \sin(\tfrac{1}{h}) \right| \leq |h| \to 0 \text{ as } h \to 0$$

Pinching Thm

$$\Rightarrow \boxed{\lim_{h \to 0} \frac{f(h) - f(0)}{h}} = f'(0) \text{ exists} \quad (\text{i.e. differentiable at } 0) \;\#$$
$$= 0$$

(ii) $$f'(x) = \begin{cases} 2x \cdot \sin(\frac{1}{x}) + x^2 \cos(\frac{1}{x}) \cdot \frac{-1}{x^2} = 2x \cdot \sin(\frac{1}{x}) - \cos(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & x = 0 \end{cases}$$

Since when $x = \frac{1}{2n\pi}$, $f'(\frac{1}{2n\pi}) = \underline{-1} \not\to 0$



we have

$$\underline{\lim_{x \to 0} f'(x) \neq 0 = f'(0)}$$

That is, $\underline{f' \text{ is NOT Continuous at } x=0}$

$\Rightarrow$ f' is NOT differentiable at $x=0$ #

5 ~~4~~. Differentiate the following functions.

Q4 is on page 10.

(a) (8 points) $f(x) = (1 + \sqrt[3]{x})^5$.

(b) (8 points) $f(x) = 3\cos x - 4\sec x$.

(c) (8 points) $f(x) = \cos(\sqrt{x})$, $x > 0$.

(a) $f'(x) = 5(1+\sqrt[3]{x})^4 \cdot \frac{1}{3}x^{-\frac{2}{3}}$

(b) $f'(x) = -3\sin x - 4\frac{-(-\sin x)}{\cos^2 x} = -3\sin x - 4\tan x \sec x$

(c) $f'(x) = -\sin(\sqrt{x}) \cdot \frac{1}{2}x^{-\frac{1}{2}}$

(d) (8 points) $f(x) = \sqrt{\sin x \cos x}$, $0 < x < \pi/2$.

(e) (8 points) $f(x) = (x+1)^{1/3}(x+2)^{2/3}$.

(d) $f'(x) = \frac{1}{2}(\sin x \cos x)^{-\frac{1}{2}} \cdot (\cos x \cdot \cos x + \sin(-\sin x))$

(e) $f'(x) = \frac{1}{3}(x+1)^{-\frac{2}{3}}(x+2)^{\frac{2}{3}} + \frac{2}{3}(x+1)^{\frac{1}{3}}(x+2)^{-\frac{1}{3}}$

$$\frac{d(2y^2)}{dx} = \underbrace{\frac{d\,2y^2}{dy}}_{4y} \cdot \frac{dy}{dx}$$

$$(xy)' = \overset{1}{(x)'} \cdot y + x \cdot y'$$

6 ~~5~~. (15 points) Find equations for the tangent line at the point indicated:

$$x^2 + xy + 2y^2 = 28, \qquad (-2, -3).$$

$\frac{d}{dx}$(both sides) $\xrightarrow{\frac{d}{dx}}$ $\frac{d}{dx}(x^2+xy+2y^2)$

$$\underline{2x + y + x\cdot\frac{dy}{dx} + 4y\cdot\frac{dy}{dx}} = 0 = \frac{d}{dx}(28)$$

$$\Rightarrow \quad \left.\frac{dy}{dx}\right|_{(-2,-3)} = -\left.\frac{2x+y}{x+4y}\right|_{(-2,-3)} = -\frac{-7}{-14}$$

$$= -\frac{1}{2}$$

$\Rightarrow$ tangent line is

$$y+3 = -\frac{1}{2}(x+2) \qquad \#$$

7 ~~6~~. (15 points) Show that

$$\tan x > x,$$

for all x in $(0, \frac{\pi}{2})$. $0 \notin (0, \frac{\pi}{2})$

Let $f(x) = \tan x - x$, $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$

$\Rightarrow f(0) = 0 - 0 = 0$, and

$f'(x) = \sec^2 x - 1 > 0 \quad \forall x \in (0, \frac{\pi}{2})$

Let x be an arbitrary point in $(0, \frac{\pi}{2})$.

Since $f(x)$ is differentiable on $(0, x)$ and continuous on $\underline{[0, x]}$, by Mean Value Thm, $\exists\, c \in (0, x)$ s.t.

$$f(x) - f(0) = \underset{\tan x - x}{\underset{\shortparallel}{f(x)}} = \overset{>0}{(x-0)} \cdot \overset{>0}{f'(c)} > 0$$

So

$$\tan x - x \quad \forall x \in (0, \frac{\pi}{2}) \qquad \#$$

8 ~~7~~. (10 points) Suppose a function f has derivative

$$f'(x) = x(x-1)^2(x+1)^3(x-2)^4.$$

At what numbers x, if any, does f have a local maximum? A local minimum?

Critical points: $x = -1, 0, 1, 2$

x		-1		0		1		2	
f'	+		−		+		+		+
f	↗	●	↘	●	↗		↗		↗

f has a local maximum at $x = -1$

and a local minimum at $x = 0$ #

9 ~~8~~. (15 points) Determine A and B so that the curve

$$y = A\cos 2x + B\sin 3x$$

has a point of inflection at $(1, 4)$.

$$y' = -2A\sin 2x + 3B\cos 3x$$

$$y'' = -4A\cos 2x - 9B\sin 3x$$

$$\begin{cases} 4 = A\cos 2 + B\sin 3 \\ 0 = -4A\cos 2 - 9B\sin 3 \end{cases} \Rightarrow \begin{cases} A = \dfrac{36}{5\cos 2} \\ B = -\dfrac{16}{5\sin 3} \end{cases}$$

#

10 ~~9.~~ Find the critical points. Then find and classify all the extreme values.

(a) (15 points) $f(x) = x^2 - 4x + 1, \quad 0 \le x \le 3.$

(b) (15 points) $f(x) = \begin{cases} x^2 + 2x + 2, & x < 0, \\ x^2 - 2x + 2, & 0 \le x \le 2. \end{cases}$

(a) $f'(x) = 2x - 4$

$\overset{\text{global max.}}{f(0) = 1}, \quad \overset{\text{global min.}}{f(2) = -3}, \quad \overset{\text{local max.}}{f(3) = -2}$

critical point: $x = 2$

x	0		2		3
f'		−		+	
f		↘		↗	

(b) $f'(x) = \begin{cases} 2x + 2 & x < 0 \\ \text{doesn't exist.} & x = 0 \\ 2x - 2 & 0 < x < 2 \end{cases}$

$f(-1) = 1$ ← global min.

$f(0) = 2$ ← local max.

$f(1) = 1$ ← global min.

$f(2) = 2$ ← local max.

critical points: $-1, 0, 1$

x		-1		0		1		2
f'	−		+		−		+	
f	↘		↗		↘		↗	

4. ~~10~~. True or false? (No need to explain.)

T (a) (2 points) If a function f is differentiable at c, then f is continuous at c.

F (b) (2 points) If a function f is continuous at c, then f is differentiable at c.

T (c) (2 points) If a function f is continuous on $[a, b]$, then f is integrable on $[a, b]$.

F (d) (2 points) If a function f is integrable on $[a, b]$, then f is continuous on $[a, b]$.

T (e) (2 points) If a function f is differentiable on $(-\infty, \infty)$, then f integrable on $[0, 1]$.

11. (10 points) Prove that the equation $6x^4 - \overset{8}{\cancel{7}}x + 1 = 0$ has exactly 2 real roots.

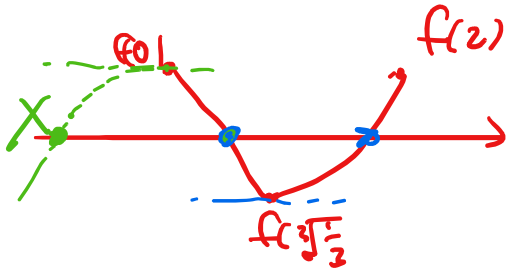
Let $f(x) = 6x^4 - 8x + 1$

$f'(x) = 24x^3 - 8$, $f'(x) = 0 \Leftrightarrow x = \sqrt[3]{\frac{1}{3}}$

$f(\sqrt[3]{\tfrac{1}{3}}) = \overset{2}{\cancel{6}}\cdot\sqrt[3]{\frac{1}{\underset{3}{\cancel{81}}}} - 8\cdot\sqrt[3]{\frac{1}{3}} + 1 = -6\sqrt[3]{\frac{1}{3}} + 1 < 0$

$f(0) = 1 > 0$

$f(2) = 6\cdot 16 - 8\cdot 2 + 1 > 0$



$\Rightarrow$ By Intermediate Value Thm, $\exists\ r_1 \in (0, \sqrt[3]{\tfrac{1}{3}})$, $r_2 \in (\sqrt[3]{\tfrac{1}{3}}, 2)$

s.t. $f(r_1) = 0 = f(r_2)$

Assume $\exists\ \underline{r_3 \neq r_1, r_2}$ s.t. $f(r_3) = 0$, There are 3 cases:

case1: $r_3 < r_1 < r_2$ case2: $r_1 < r_3 < r_2$ case 3: $r_1 < r_2 < r_3$

We prove case1, and the other 2 cases are similar:

By Mean Value Thm, $\exists$ $c_1 \in (r_3, r_1)$, $c_2 \in (r_1, r_2)$ s.t.

$$f'(c_1) = \frac{\overset{0}{f(r_3)} - \overset{0}{f(r_1)}}{r_3 - r_2} = 0 \quad \text{and} \quad c_1 \neq c_2$$

$$f'(c_2) = \frac{f(r_2) - f(r_1)}{r_2 - r_1} = 0$$

This is a contradiction because $f'(x) = \underline{24x^3 - 8 = 0}$ has one and only one real root. So $f(x) = 0$ has exactly 2 real roots. #

$\lim_{x\to 0} f(x) = f(0)$ $\lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$ exists

3. (麟翔)
許多學生對分段函數不甚了解。有一部分學生把在 0 連續的定義當成在 0 可微的定義。也有部分學生直接對 $x^2 \sin(1/x)$ 直接微分然後再把 x 逼近 0 說他微分不存在。

6. (登科)
有不少學生算錯 xy 的微分，大部分應該不知道在這裡要用到 chain rule 和 product rule。

7. (登科)
大部分的學生使用 $f(x) = \tan x - x$ 的嚴格遞增來證明。大家的論述最後都是 $x > 0$ 所以 $f(x) > f(0)$，但是 0 沒有在定義域中。

10.(b) (俊碩)
大部分的同學沒有指出 $f'(x)$ 在 $x = 0$ 不存在，進而寫下

$$f'(x) = \begin{cases} 2x+2, & x < 0 \\ 2x-2, & 0 \le x \le 2 \end{cases}$$

另外有一部分同學指出 $f'(0)$ 不存在的理由是微分後在零點不連續:

$$\lim_{h\to 0} \frac{f(0+h)-f(0)}{h} \qquad \lim_{x\to 0^+} f'(x) \neq \lim_{x\to 0^-} f'(x)$$

doesn't exist

11. (俊碩)
許多同學的理由是 $f(\sqrt[3]{\frac{1}{3}}) < 0$，然後說在左側遞減右側遞增，因此會交於軸 $y = 0$。若同學有特別說明凹口向上才會正確，否則論證是不足的。對於至多兩根同學部分使用均值定理，有一部分同學沿用前面嚴格遞增與遞減來說穿過 $y = 0$ 後將不再會回頭穿過第三次，雖然是正確的，但因為同學前面若沒有保證有根，就無法得出此結論。因此本題扣分較為嚴重的原因大多是前面已經沒有說明根的存在性。

$g(x) = -\frac{1}{x}$