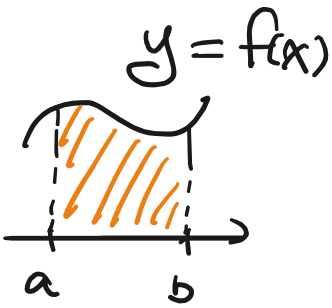
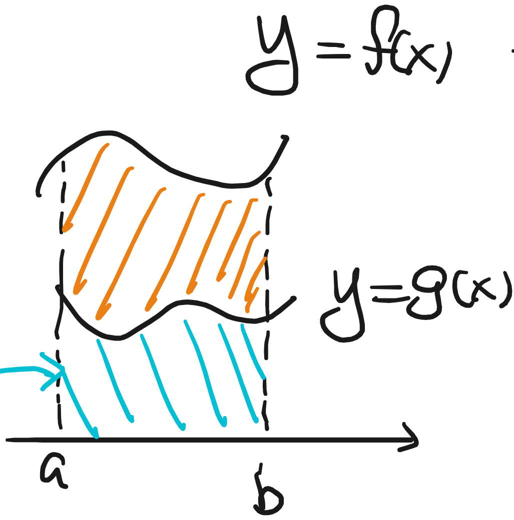
Calculus 11/28

Applications of integration

Area:

Area of 

$= \int_a^b f(x)\,dx$

Area of 

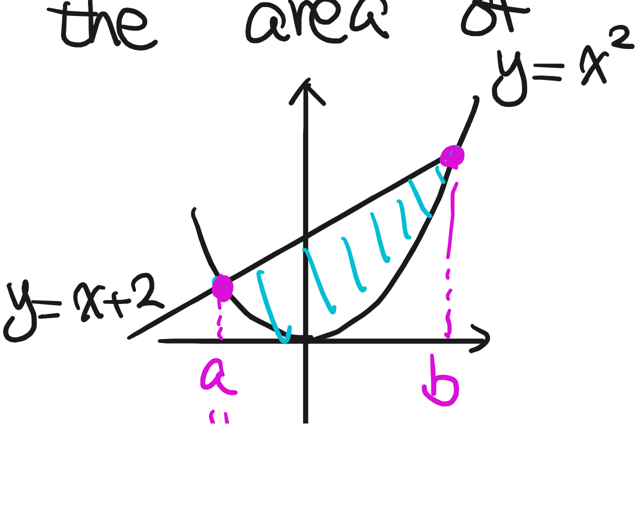
$\int_a^b g(x)\,dx$ →

(orange ////) + (blue \\\\) $= \int_a^b f(x)\,dx$

> Area (////)
> $= \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$
> $= \int_a^b f(x) - g(x)\,dx$

Example

① Find the area of



Sol

Step1 Find a and b:

Solve $\begin{cases} y = x^2 \\ y = x+2 \end{cases}$ $\Rightarrow$ $x^2 = x+2$ $\Rightarrow$ $x^2 - x - 2 = 0$

$= (-2) \times 1$

$\Rightarrow (x-2)(x+1) = 0 \Rightarrow x = -1, 2$

Step2: Find the area

$$\text{Area} = \int_{-1}^{2} (x+2) - x^2 \, dx$$

$(2x)'$, $\left(-\frac{x^3}{3}\right)'$, $\left(\frac{x^2}{2}\right)'$

fundamental thm of Calculus

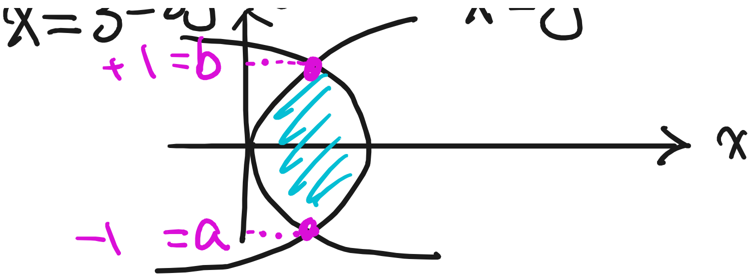
$$= \frac{x^2}{2} + 2x - \frac{x^3}{3} \Big|_{-1}^{2}$$

$$= \frac{4}{2} + 4 - \frac{8}{3} - \left(\frac{1}{2} + (-2) - \frac{-1}{3}\right)$$

$$= 8 - \frac{9}{3} - \frac{1}{2} = \frac{9}{2} \quad \#$$

② Find the area of

$x = 2 - 2y^2 - y$ $\quad x = y^2$



Sol

Step1: Find a and b:

Solve $\begin{cases} x = y^2 \\ x = 3-2y^2 \end{cases}$ $\Rightarrow y^2 = 3-2y^2 \Rightarrow 3y^2 = 3 \Rightarrow y = \pm 1$

Step2

$$\text{Area} = \int_{-1}^{1} (3-2y^2) - y^2 \, dy$$

$$= \int_{-1}^{1} 3 - 3y^2 \, dy \quad \left(3 = (3y)', \; 3y^2 = (y^3)'\right)$$

$$= 3y - y^3 \Big|_{-1}^{1}$$

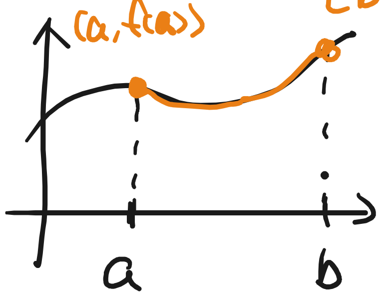
$$= (3-1) - (-3-(-1))$$

$$= 4 \; \#$$

Arc length

If f' is continuous on $[a,b]$,

then the length of $y=f(x)$



$$= \int_a^b \sqrt{1+(f'(x))^2}\, dx$$

Example

Find the length of the graph of

$$y = \frac{2\sqrt{2}}{3} x^{\frac{3}{2}} - 1, \qquad 0 \le x \le 1$$

Sol

ans $= \int_0^1 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$

$= \frac{2\sqrt{2}}{3} \cdot \frac{3}{2} x^{\frac{1}{2}} = \sqrt{2}\, x^{\frac{1}{2}}$

Let $u = 1+2x$

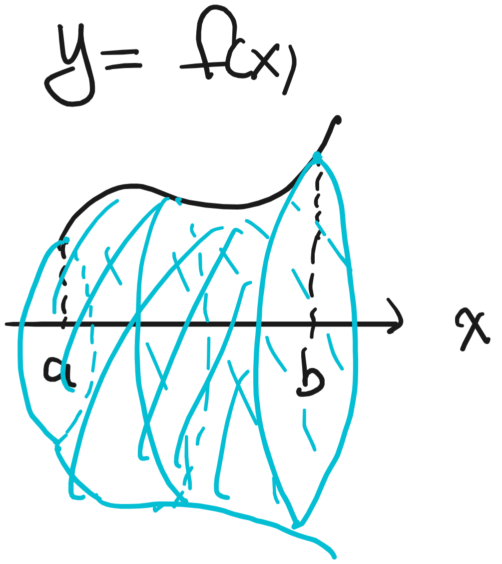
$$= \int_0^1 \sqrt{1+2x}\, dx \quad \Rightarrow du = 2dx$$

integration by substitution

$$\overset{\downarrow}{=} \int_{u(0)=1}^{u(1)=3} \sqrt{u}\ \frac{1}{2}\, du = \frac{2}{3} u^{\frac{3}{2}} \cdot \frac{1}{2} \Big|_1^3$$

$$= \frac{1}{3} 3\sqrt{3} - \frac{1}{3} 1 = \sqrt{3} - \frac{1}{3} \quad \#$$

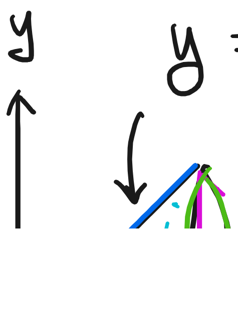
Area of the surface generated by revolving



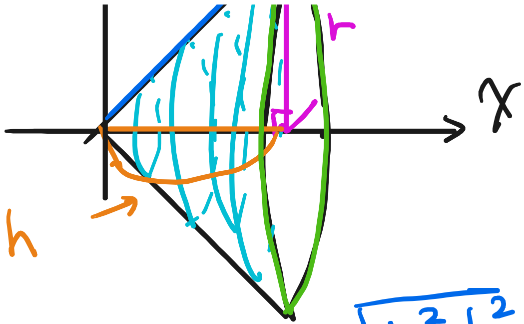
$$\text{Area} = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Example

$$y = \frac{r}{h} x$$

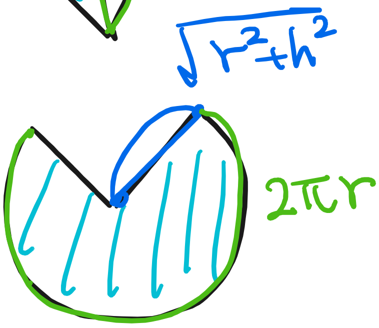


Area = ?



<u>Sol</u>

<u>Method I</u>:



$$\text{Area} = \pi\left(\sqrt{r^2+h^2}\right)^2 \cdot \frac{2\pi r}{2\pi\sqrt{r^2+h^2}}$$

$$= \pi r\sqrt{r^2+h^2}$$

#

<u>Method II</u>:

$$\left(y = \frac{r}{h}x \Rightarrow \frac{dy}{dx} = \frac{r}{h} \right)$$

$$\text{Area} = \int_0^h 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\ dx$$

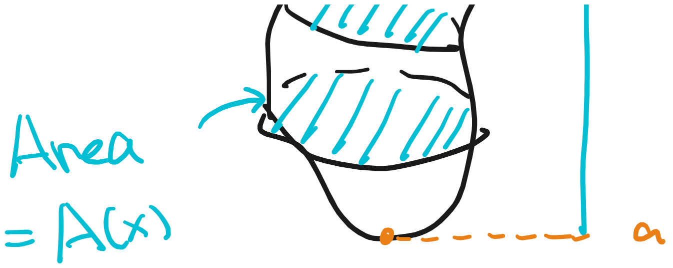
$$= \int_0^h 2\pi\,\frac{r}{h}x\sqrt{1+\left(\frac{r}{h}\right)^2}\ dx$$

$$= \pi\frac{r}{h}\sqrt{1+\left(\frac{r}{h}\right)^2}\ x^2\Big|_0^h$$

$$= \pi\frac{r}{h}\sqrt{1+\frac{r^2}{h^2}}\cdot h^2 = \pi r\sqrt{h^2+r^2}$$ #

<u>Volume</u>

Volume

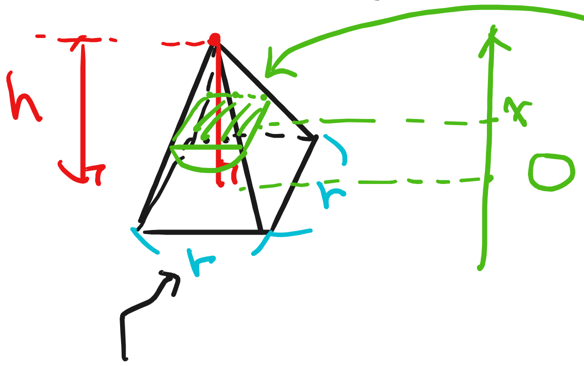


Volume

$$\Rightarrow \quad = \int_a^b A(x)\,dx$$

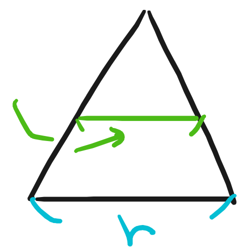
Example

Find the volume of



Area = A(x)

$$= \left(\frac{h-x}{h}\cdot r\right)^2$$



$$\Rightarrow \text{volume} = \int_0^h A(x)\,dx$$

$$= \int_0^h \left(\frac{h-x}{h}\, r\right)^2 dx$$

$$\left(-\frac{(h-x)^3}{3}\right)'$$

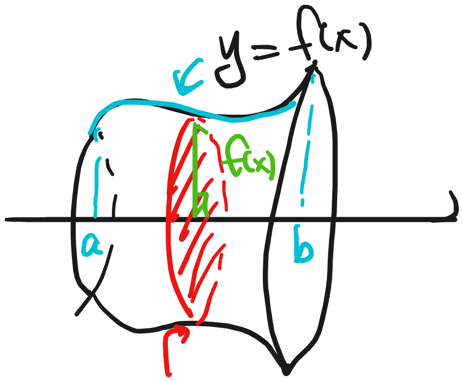
$$= \int_0^h \frac{r^2}{h^2}(h-x)^2\,dx$$

$$= -\frac{r^2}{h^2}\frac{(h-x)^3}{3}\Big|_0^h$$

$$= \frac{r^2}{h^2}\frac{h^3}{3} = \frac{r^2 h}{3} \quad \#$$

Solid of revolution (§6.2)

Volume of



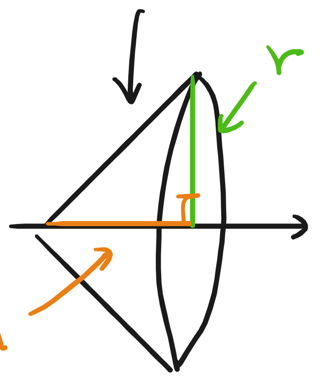
$A(x) = \pi (f(x))^2$

$$\text{Volume} = \int_a^b A(x)\,dx = \int_a^b \pi (f(x))^2\,dx$$

Example

①

$y = \frac{r}{h} x$



Volume = ?

Sol

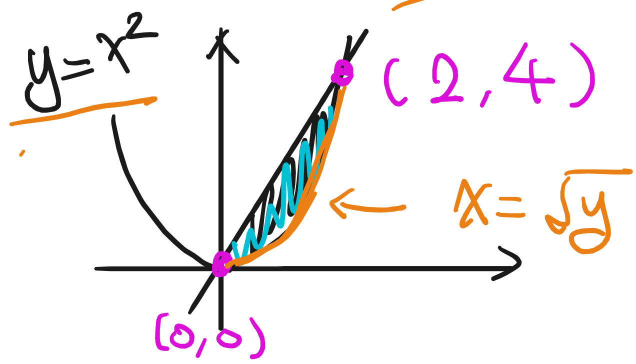
$$\text{Volume} = \int_0^h \pi \left(\frac{r}{h} x\right)^2 dx$$

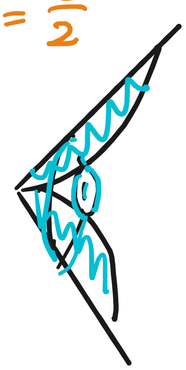
$$= \pi \frac{r^2}{h^2} \int_0^h x^2\,dx$$

$$\frac{r^2}{} \left[x^3 \right|^h = \pi\, r^2$$

$= \pi \frac{r^2}{h^2} \frac{1}{3} x^3 \Big|_0^h = \frac{\pi}{3} r^2 h$ #

② Find the volume the solid generated by revolving $y = 2x \Leftrightarrow x = \frac{y}{2}$





(i) about x-axis, and

(ii) about y-axis

sol

(i) $= \int_0^2 \pi (2x)^2 - \pi (x^2)^2 \, dx$

$= \int_0^2 4\pi x^2 - \pi x^4 \, dx$

$= \frac{4\pi}{3} x^3 - \frac{\pi}{5} x^5 \Big|_0^2$

$= \frac{4\pi}{3} 2^3 - \frac{\pi}{5} 2^5 = \frac{64\pi}{\;}$

$= \frac{\quad}{3}\cdot 8 - \frac{\quad}{5}\cdot 32 = \frac{\quad}{15}$ #

(ii) $= \int_0^4 \pi(\sqrt{y})^2 - \pi\left(\frac{y}{2}\right)^2 \, dy$

$= \int_0^4 \pi y - \frac{\pi}{4} y^2 \, dy$

$= \frac{\pi}{2} y^2 - \frac{\pi}{4} \frac{y^3}{3} \Big|_0^4$

$= 8\pi - \frac{16}{3}\pi = \frac{8}{3}\pi$ #