Calculus 1/23

Thm (§8.2, Integration by parts 分部積分)

Suppose u and v are functions with continuous derivatives.

Then

$$\int u(x)\,\underbrace{v'(x)\,dx}_{dv} = u(x)\cdot v(x) - \int v(x)\cdot \underbrace{u'(x)\,dx}_{du}$$

and

$$\boxed{\int_a^b u(x)\,v'(x)\,dx = u(x)\cdot v(x)\Big|_a^b - \int_a^b v(x)\,u'(x)\,dx}$$

Notation: Using $u'(x)\,dx = du$, we have

$$\int u\,dv = uv - \int v\,du$$

pf

Recall

$$(u\cdot v)' = u'\cdot v + u\cdot v'$$

$$\Rightarrow \underbrace{\int (u\cdot v)'\,dx}_{u\cdot v + C} = \int u'\cdot v\,dx + \int u\cdot v'\,dx$$

there's another constant here

$$\Rightarrow \int u\cdot v'\,dx = u\cdot v + C - \int v\cdot u'\,dx$$

combine

$$= u\cdot v - \int v\cdot u'\,dx \quad \#$$

Example

Let $u(x) = x \Rightarrow u'(x) = 1$

$v'(x) = \sin x$

$v(x) = -\cos x$

① $\int_0^{\pi} \underbrace{x}_{u}\, \overbrace{\sin x}^{v'}\, dx = ?$

$$\underset{\text{Thm}}{=} u(x)v(x)\Big|_0^{\pi} - \int_0^{\pi} v(x)\,u'(x)\,dx$$

$$= x\cdot(-\cos x)\Big|_0^{\pi} - \int_0^{\pi} (-\cos x)\cdot 1\,dx$$

$$= -x\cos x\Big|_0^{\pi} + \sin x\Big|_0^{\pi}$$

$$= -\pi\underbrace{\cos\pi}_{-1} - \underbrace{(-0\cdot\cos 0)}_{=0} + \underbrace{\sin\pi}_{=0} - \underbrace{\sin 0}_{=0}$$

$= \pi \quad \#$

② $\int_0^1 x\sqrt{x+1}\,dx = ?$

Let
$u(x) = x \Rightarrow u'(x) = 1$
$v'(x) = \sqrt{x+1} \Leftarrow v(x) = \frac{2}{3}(x+1)^{\frac{3}{2}}$
$\| \quad (x+1)^{\frac{1}{2}} = \left(\frac{2}{3}(x+1)^{\frac{3}{2}}\right)'$

$= \int_0^1 u \cdot v'\,dx$

$= u \cdot v \Big|_0^1 - \int_0^1 v \cdot u'\,dx$

$= x \cdot \frac{2}{3}(x+1)^{\frac{3}{2}} \Big|_0^1 - \int_0^1 \frac{2}{3}(x+1)^{\frac{3}{2}} \cdot 1\,dx$

$1 \cdot \frac{2}{3} \cdot (1+1)^{\frac{3}{2}} - 0 \cdot \frac{2}{3} \cdot (0+1)^{\frac{3}{2}} = \frac{2}{3} \cdot 2\sqrt{2} - 0 = \frac{4}{3}\sqrt{2}$

$= \frac{4}{3}\sqrt{2} - \int_0^1 \frac{2}{3}(\underset{w}{x+1})^{\frac{3}{2}}\,dx$

$dw = (x+1)' \cdot dx$
$= dx$

$\left(\frac{2}{3} \cdot \frac{2}{5}(x+1)^{\frac{5}{2}}\right)'$

$= \frac{4}{3}\sqrt{2} - \int_{w(0)=1}^{w(1)=2} \frac{2}{3} w^{\frac{3}{2}}\,dx$

$= \left(\frac{2}{5} \cdot w^{\frac{5}{2}}\right)'$

$4 \quad 5 \quad |^2$

$$= \frac{4}{3}\sqrt{2} - \frac{\text{[illegible]}}{15} w^2 \Big|_1$$

$$= \frac{4}{3}\sqrt{2} - \frac{16}{15}\sqrt{2} + \frac{4}{15}$$

$$= \frac{4}{15}\sqrt{2} + \frac{4}{15} \quad \#$$

More properties:

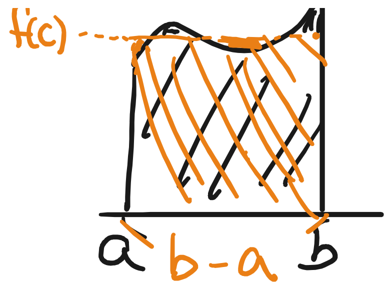
Thm (First mean value theorem for integral
Thm 5.9.1)

If f is continuous on $[a,b]$,
then $\exists$ $c \in [a,b]$ s.t.

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

called the average value or mean value of $f(x)$ on $[a,b]$

pf



Let

$$F(x) = \int_a^x f(t)\,dt \leftarrow \text{differentiable on } (a,b)$$

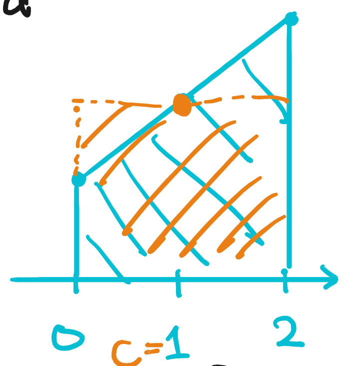
$f = F'$ continuous on $[a,b]$

By Mean Value Thm, $\exists\, c \in (a,b)$ s.t.

$$F'(c) = \frac{F(b)-F(a)}{b-a} = \frac{\int_a^b f(t)\,dt - 0}{b-a}$$

$$= \frac{1}{b-a}\int_a^b f(x)\,dx$$

#



Example

$f(x) = x+1$

$a=0,\ b=2$

$$\Rightarrow \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{2-0}\int_0^2 x+1\,dx$$

$= \left(\frac{x^2}{2}+x\right)'$

$$= \frac{1}{2}\left(\frac{x^2}{2}+x\right)\Big|_0^2 = 2$$

$$= f(1) = 1+1$$

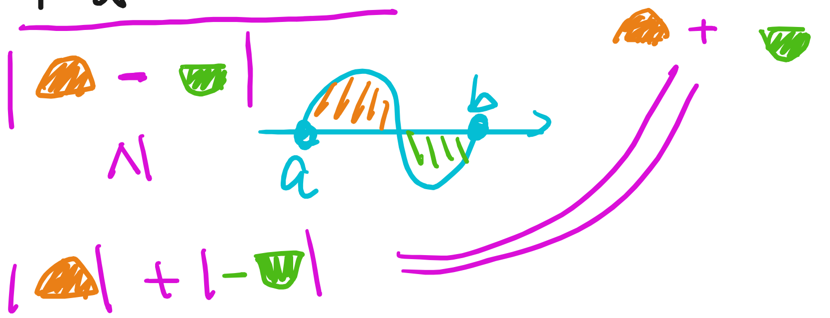
"c"

Thm (§5.8)

S f and g are integrable.

Suppose f and g are

(i) $$\left|\int_a^b f(x)\,dx\right| \leqslant \int_a^b |f(x)|\,dx$$



(ii) If f is continuous on [a, b],

then

$$\frac{d}{dx}\int_a^{u(x)} f(t)\,dt = f(u(x))\cdot u'(x)$$

chain rule + $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$

奇函數

(iii) If f is an odd function on [-a, a]

i.e. $f(-x) = -f(x)$ $\forall x \in [-a, a]$,

e.g. $\sin x$: $\sin(-x) = -\sin x$

$f(x) = x^3 \Rightarrow (-x)^3 = -x^3$

then

$$\int_{-a}^{a} f(x)\,dx = 0$$

pf

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx$$

$-a$

Let $u = -x$

$du = -dx$

$$= \int_{u(-a)=a}^{u(0)=0} \boxed{f(-u)}\cdot(-1)\,du + \int_0^a f(x)\,dx$$

odd function

$= -f(u)$

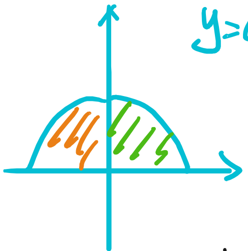
$$= -\int_0^a f(u)\,du + \int_0^a f(x)\,dx$$

$$= 0 \qquad \#$$

偶函數

(iv) If f is an even function on $[-a,a]$

i.e. $f(-x) = f(x) \quad \forall x \in [-a,a]$



y=cosx e.g. • $\cos x$: $\cos(-x) = \cos x$

• x^2 : $(-x)^2 = x^2$

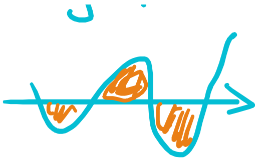
then

$$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx.$$

(v) If f is continuous on $[a,b]$

and if

y = f(x) $\int^b$



$$\int_a |f(x)|\, dx = 0$$

then $f(x) = 0 \quad \forall x \in [a,b]$

<u>Example</u>

① $\dfrac{d}{dx}\left(\int_x^{x^2} \sin(t^2)\, dt\right)$

$$= \frac{d}{dx}\left(\int_0^{x^2} \sin(t^2)\, dt + \int_x^0 \sin(t^2)\, dx\right)$$

$$= \frac{d}{dx}\left(\int_0^{x^2} \sin(t^2)\, dt\right) - \frac{d}{dx}\left(\int_0^{x} \sin(t^2)\, dt\right)$$

$$\overset{(ii)}{=} \sin\left((x^2)^2\right) \cdot (x^2)' - \sin x^2$$

$$= 2x \sin(x^4) - \sin(x^2) \quad \#$$

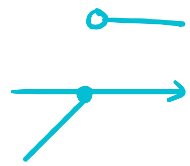
(iii)

② $\displaystyle\int_{-1}^{1} x^{1}\,dx \overset{(iii)}{=} 0$ #

③ $\displaystyle\int_{-1}^{1} x^{2}\,dx \overset{(iv)}{=} 2\int_{0}^{1} x^{2}\,dx$

$\displaystyle = 2\,\frac{x^3}{3}\Big|_0^1 = \frac{2}{3}$ #

④ $f(x) = \begin{cases} 1 & , x > 0 \\ x & , x \le 0 \end{cases}$



$\displaystyle\int_{-1}^{1} f(x)\,dx = ?$

$\displaystyle = \int_{-1}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx$

$\displaystyle = \int_{-1}^{0} x\,dx + \int_{0}^{1} 1\,dx$

$\displaystyle = \frac{x^2}{2}\Big|_{-1}^{0} + x\Big|_0^1$

$\displaystyle = -\frac{1}{2} + 1 = \frac{1}{2}$ #