Calculus 11/21

Recall

Suppose f is continuous on $[a,b]$

and

$$G'(x) = f(x) \qquad \forall x \in (a,b)$$

Then

$$\boxed{\int_a^b f(x)\,dx = G(b) - G(a)}$$

Example

① (i) Suppose

$$F(x) = \int_0^{x} \sin(\pi t)\,dt.$$

Q: $F'(\frac{3}{4}) = ?$

A: $F'(\frac{3}{4}) = \sin(\pi x)\Big|_{x=\frac{3}{4}}$

$$= \sin(\tfrac{3}{4}\pi) = \frac{\sqrt{2}}{2}$$

#

(ii)

Suppose

$$G(x) = \int_0^{x^2} \sin(\pi t)\, dt.$$

Q: $G'(\frac{3}{4}) = ?$

A: Since $G(x) = F(x^2)$,

(chain rule) $G'(x) = F'(x^2) \cdot (x^2)'$

$$= \sin(\pi x^2) \cdot 2x$$

$$\Rightarrow G'(\tfrac{3}{4}) = \sin(\pi \cdot \tfrac{9}{16}) \cdot \frac{3}{2} \quad \#$$

② $\displaystyle\int_1^4 x^2\, dx = \int_1^4 \left(\frac{x^3}{3}\right)' dx$

Since $\left(\dfrac{x^3}{3}\right)' = \dfrac{3x^2}{3} = x^2$

$$= \frac{x^3}{3}\Big|_1^4 = \frac{4^3}{3} - \frac{1^3}{3}$$

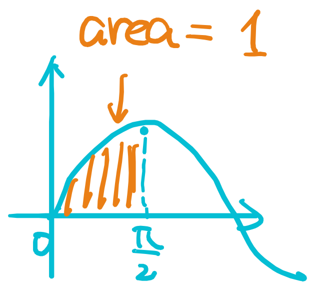
$$= \frac{64-1}{3} = 21 \quad \#$$

$(-\cos x)' = \sin x$

③ $$\int_0^{\frac{\pi}{2}} \sin x \, dx \;\overset{\downarrow}{=}\; \int_0^{\frac{\pi}{2}} (-\cos x)' \, dx$$

$$= -\cos x \Big|_0^{\frac{\pi}{2}} = -\cos\frac{\pi}{2} - (-\cos 0)$$

$$= 0 + 1 = 1 \quad \#$$



④ $$\int_0^1 2x - 6x^4 + 5 \, dx$$

$$= \int_0^1 \left(x^2 - \frac{6}{5}x^5 + 5x \right)' dx$$

$(x^n)' = n x^{n-1}$

$$= x^2 - \frac{6}{5}x^5 + 5x \Big|_0^1$$

$$= 1^2 - \frac{6}{5}1^5 + 5 = 6 - \frac{6}{5} = \frac{24}{\ }$$

5 #

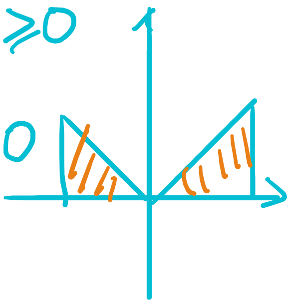
(5) $\int_0^1 x^{\frac{3}{2}} - x^{\frac{1}{2}}\, dx$

$= \int_0^1 \left(\frac{2}{5} x^{\frac{5}{2}} - \frac{2}{3} x^{\frac{3}{2}}\right)' dx$

$= \frac{2}{5} x^{\frac{5}{2}} - \frac{2}{3} x^{\frac{3}{2}} \Big|_0^1$

$= \frac{2}{5} - \frac{2}{3} = \frac{-4}{15}$ #

(6) $\int_{-1}^1 |x|\, dx$ $\quad |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$



$= \int_{-1}^0 -x\, dx + \int_0^1 x\, dx$

$= -\frac{x^2}{2}\Big|_{-1}^0 + \frac{x^2}{2}\Big|_0^1$

$(-1)^2$ 1^2

$$= 0 - \left(-\frac{1}{2}\right) + \frac{1}{2} - 0$$

$$= \frac{1}{2} + \frac{1}{2} = 1 \quad \#$$

$$2\sec x \tan x - 5\sec^2 x$$

$$\text{⑦} \quad \int_0^{\frac{\pi}{4}} \underbrace{\sec x \cdot (2\tan x - 5\sec x)}_{} \, dx$$

$$= 2\int_0^{\frac{\pi}{4}} \sec x \tan x \, dx - 5\int_0^{\frac{\pi}{4}} \underbrace{\sec^2 x}_{= (\tan x)'} \, dx$$

recall

$$(\sec x)' = \left(\frac{1}{\cos x}\right)' \quad \text{pf}$$

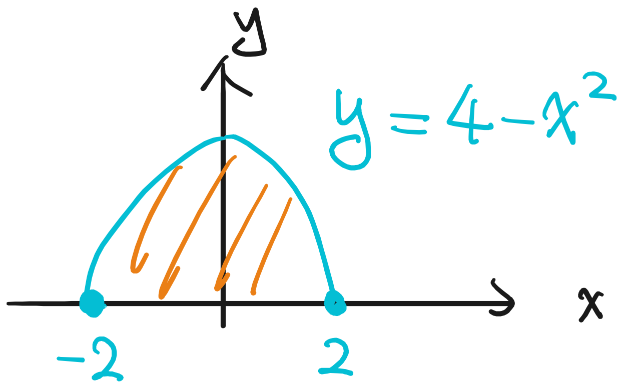
$$= \frac{-(\cos x)'}{\cos^2 x} = \frac{\sin x}{\cos^2 x} = \sec x \tan x$$

$$= 2 \sec x \Big|_0^{\frac{\pi}{4}} - 5 \tan x \Big|_0^{\frac{\pi}{4}}$$

$$= 2(\sqrt{2} - 1) - 5(1 - 0)$$

$= 2\sqrt{2} - 1$ #

⑧ Find the area



Sol

Step1: Solve $0 = 4 - x^2$

$\Rightarrow x = \pm 2$

Step2:

$$\text{area} = \int_{-2}^{2} 4 - x^2 \, dx$$

$$= 4x - \frac{x^3}{3} \Big|_{-2}^{2}$$

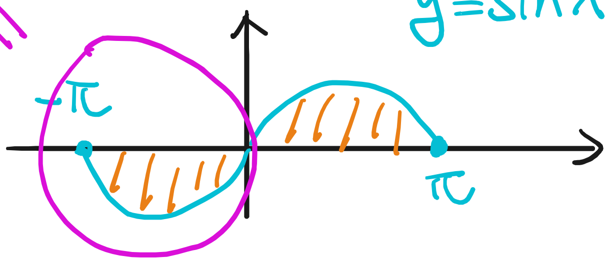
$$= \left(8 - \frac{8}{3}\right) - \left(-8 - \frac{8}{3}\right)$$

$$= 2 \cdot \frac{2}{3} \cdot 8 = \frac{32}{3}$$ #

⑨ Find the area

$-\int_{-\pi}^{0} \sin x \, dx$

$\parallel$



NOTE:

$$\int_{-\pi}^{\pi} \sin x \, dx$$

$$= -\cos x \Big|_{-\pi}^{\pi} = 0$$

<u>sol</u>

$$-\sin x$$
$$\parallel$$

$$\text{Area} = \int_{-\pi}^{0} |\sin x| \, dx + \int_{0}^{\pi} \sin x \, dx$$

$$= \cos x \Big|_{-\pi}^{0} + (-\cos x)\Big|_{0}^{\pi}$$

$$= 1 - (-1) + \big(-(-1) - (-1)\big)$$

$$= 4 \quad \#$$

<u>Remark</u>

People write

$$\int f(x)\,dx = F(x) + C$$

if $F'(x) = f(x)$.

For example,

- $\int x\,dx = \frac{1}{2}x^2 + C$

  means: $\left(\frac{1}{2}x^2\right)' = x$

- $\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$

- $\int \sin x\,dx = -\cos x + C$

<u>Integration by substitution</u> (a.k.a. u-substitution)

<u>Thm</u> (Thm 5.7.1, (5.7.2))

If f is continuous, $F' = f$,

and u is differentiable, then

$$\int \overset{=F'}{f}(u(x))\cdot u'(x)\,dx = F(u(x)) + C$$

and

$$\int_a^b f(u(x))\cdot u'(x)\,dx$$

$$= F(u(b)) - F(u(a))$$

$$= \int_{u(a)}^{u(b)} f(u)\,du$$

Convenient notation:

"$du = u' \cdot dx$"

$$\int f(u(x))\cdot u'(x)\,dx = \int f(u)\,du$$

Example

① $\displaystyle\int_0^1 \frac{1}{(3+5x)^2}\,\overset{\frac{1}{5}du}{\overset{\shortparallel}{dx}} = ?$

Let $u = 3+5x$

$\Rightarrow du = 5dx$

$= (3+5x)' \cdot dx$

$u(1) = 8$

$$= \int_{u(0)=3}^{u(1)} \frac{1}{u^2} \frac{1}{5}\, du$$

$$\frac{1}{u^2} = u^{-2} = \frac{1}{-1} u^{-1}$$

$$= -\frac{1}{5} \cdot \frac{1}{u} \Big|_3^8 = -\frac{1}{40} - \left(-\frac{1}{15}\right)$$

$$= \frac{1}{24} \quad \#$$

② $\displaystyle\int_0^1 x^2 \sqrt{4+x^3}\, dx$

$$\underbrace{x^2\,dx}_{= \frac{1}{3} du}$$

Let

$u = 4 + x^3$

$du = 3x^2 \cdot dx$

$$= \int_{u(0)=4}^{u(1)=5} \sqrt{u}\, \frac{1}{3}\, du$$

$$= \frac{1}{3} \frac{2}{3} u^{\frac{3}{2}} \Big|_4^5 = \frac{2}{9}\left(5^{\frac{3}{2}} - 4^{\frac{3}{2}}\right)$$

$$= \frac{2}{9}\left(5\sqrt{5} - 8\right) \quad \#$$

$= \frac{1}{3} du$

③ $\displaystyle\int_0^2 (x^2-1)(\underbrace{x^3-3x+2}_{\text{let } u})^3\,dx \Rightarrow du = (3x^2-3)\,dx$

$u(2) = 8-6+2 = 4$

$$= \int_{u(0)=2}^{u(2)} u^3 \, \frac{1}{3}\, du$$

$$= \frac{1}{3}\,\frac{1}{4}\,u^4 \Big|_2^4 = \frac{1}{12}(256-16)$$

$$= 20 \quad \#$$

④ $\displaystyle\int_0^{\frac{1}{2}} \cos^3(\pi x)\cdot \underbrace{\sin(\pi x)\,dx}_{= -\frac{1}{\pi}du}$

$(\cos(\pi x))' = -\sin(\pi x)\cdot\pi$

Let $u = \cos(\pi x) \Rightarrow du = -\sin(\pi x)\cdot\pi\cdot dx$

$u(\frac{1}{2}) = \cos\frac{\pi}{2} = 0$

$$= \int_{u(0)=1}^{u(\frac{1}{2})} u^3\cdot\left(-\frac{1}{\pi}\right) du = -\frac{1}{\pi}\,\frac{1}{4}\,u^4$$

$$\frac{1}{\ }u^4\Big|^0 = 0 - \left(-\frac{1}{\ }\right)$$

$$= -\frac{\quad}{4\pi} u \Big|_{1} - 0 \quad \Big(\quad \frac{\quad}{4\pi} \Big)$$

$$= \frac{1}{4\pi} \quad \#$$