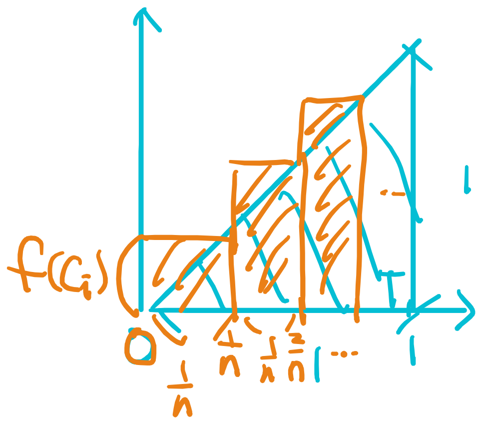
Calculus 11/14

Example

$f(x) = x$



$$\int_0^1 f(x)\,dx = \frac{1}{2}$$

(by high school math)

Suppose

$$P_n = \left\{0, \frac{1}{n}, \frac{2}{n}, \cdots, \frac{n}{n} = 1\right\}$$

$$c_i = \frac{i}{n} \in \left[\frac{i-1}{n}, \frac{i}{n}\right] \qquad f\left(\frac{i}{n}\right) = \frac{i}{n}$$

$$R_f(P) = \sum_{i=1}^{n} f(c_i) \cdot \underbrace{\left(\frac{i}{n} - \frac{i-1}{n}\right)}_{\frac{1}{n}}$$

$$= \sum_{i=1}^{n} \frac{i}{n} \cdot \frac{1}{n} = \frac{1}{n^{\cancel{2}}} \underbrace{\left(\sum_{i=1}^{n} i\right)}_{= \frac{(n+1)\cancel{n}}{2}}$$

$$= \frac{n+1}{2n} = \frac{1 + \frac{1}{n}}{2}$$

$\frac{1}{n} \to 0$ as $n \to \infty$

$$\longrightarrow \frac{1}{2} \quad \text{as} \quad n \to \infty$$

So
$$\int_0^1 x \, dx = \frac{1}{2} \qquad \#$$

<u>Example</u>

The function

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational (有理數)} \\ 0 & x \text{ is irrational (無理數)} \end{cases}$$

is NOT (Riemann) integrable

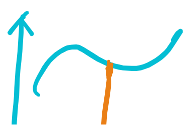
<u>Thm</u> (§5.3, §5.4)

Suppose f, g are integrable on $[a,b]$

Then

(i) $$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

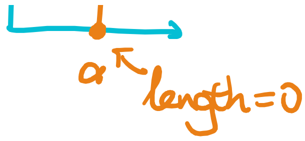
(ii) $$\int_a^a f(x)\,dx = 0$$



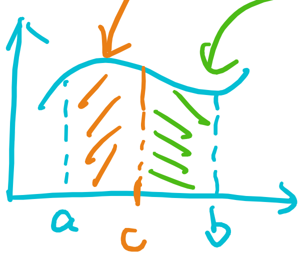
numbers

(iii) $$\int_a^b \underline{\underline{\alpha}}\cdot f(x) + \underline{\underline{\beta}}\cdot g(x)\,dx$$

$$= \alpha\cdot\int_a^b f(x)\,dx + \beta\cdot\int_a^b g(x)\,dx$$



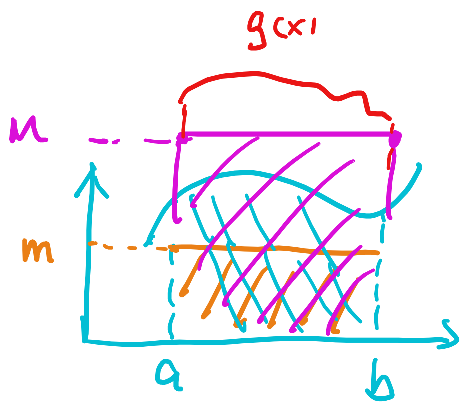
(iv) $$\int_a^b f(x)\,dx = \underbrace{\int_a^c f(x)\,dx}_{} + \int_c^b f(x)\,dx$$



(v) If $m \le f(x) \le M \quad \forall x \in [a,b]$,

then

$$\underbrace{m\cdot(b-a)}_{\int_a^b m\,dx} \le \int_a^b f(x)\,dx \le \underbrace{M\cdot(b-a)}_{\int_a^b M\,dx}$$



(vi) If $f(x) \geq g(x)$ $\forall x \in [a,b]$, then

$$\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$$

Fundamental theorem of calculus

Thm (Thm 5.3.5 , Thm 5.4.2)

Suppose f is continuous on $[a,b]$.

(i) The function

$$F(x) := \int_a^x f(t)\,dt$$

is continuous on $[a,b]$ and differentiable on (a,b).

Furthermore,

$$\boxed{\begin{aligned} F'(x) &= \frac{d}{dx}\int_a^x f(t)\,dt \\ &= f(x) \qquad \forall x \in (a,b) \end{aligned}}$$

(ii) If $G'(x) = f(x)$ $\forall x \in (a,b)$, then

☆ $$\int_a^b f(x)\,dx = \int_a^b G'(x)\,dx$$

$$= G(b) - G(a) = G(x)\Big|_{x=a}^{b}$$

notation

Such a function G is called an antiderivative of f

Example

Since

$$\left(\frac{x^3}{3}\right)' = x^2$$

by (ii),

$$\int_0^1 x^2\,dx = \frac{x^3}{3}\Big|_0^1$$

$$= \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3} \quad \#$$

Pf (Fundamental Thm of Calculus)

(i)

$$F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt$$

$$= \lim_{h\to 0} \frac{F(x+h) - F(x)}{h}$$

$$\parallel$$

$$\frac{1}{h}\left(\int^{x+h} f(t)\,dt - \int^{x} f(t)\,dt\right)$$

$h \searrow \int_a \qquad\qquad \int_a \searrow \nearrow$

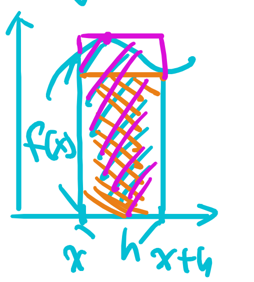
$$\int_a^x \cancel{f(t)}\,dt + \int_x^{x+h} f(t)\,dt$$

area $\sim h \cdot f(x)$

$$= \frac{1}{\cancel{h}} \underbrace{\int_x^{x+h} f(t)\,dt}$$

idea: $\sim \cancel{h} \cdot f(x)$

$\to f(x)$



(for h smooth)

Since f is continuous on $[x, x+h]$, f attains a maximum M_h at $x_{M,h} \in [x, x+h]$

and a min. m_h at $x_{m,h} \in [x, x+h]$

Since

$$m_h \le f(t) \le M_h \quad \forall t \in [x, x+h]$$

$$\underset{\substack{\| \\ f(x_{m,h})\cdot h}}{m_h \cdot (x+h-x)} \le \int_x^{x+h} f(t)\,dt \le \underset{\substack{\| \\ f(x_{M,h})\cdot h}}{M_h \cdot (x+h-x)}$$

$$\Rightarrow \quad f(x_{m,h}) \le \frac{1}{h}\int_x^{x+h} f(t)\,dt \le f(x_{M,h})$$

$\downarrow$ as $h \to 0$ $\downarrow$

$f(x)$ (by continuity) $f(x)$

So by Pinching Thm,

$$F'(x) = \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)\,dt = f(x)$$ #

(ii) Assume $G'(x) = f(x) = F'(x)$

Then $G(x) = F(x) + C$

for some constant C.

So

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt - \int_a^a f(t)\,dt$$

(0 =)

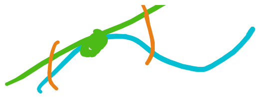
$$= F(b) - F(a)$$

$$= (G(b) - \cancel{C}) - (G(a) - \cancel{C})$$

$$= G(b) - G(a)$$ #

$y = 3x + 2)$

HW6.5. Q6:



$y = f(x)$

Assume $y = f(x)$ $\quad y = 3x+2$

$$\circledast \quad |f(x) - (3x+2)| \leq |x|^{\frac{3}{2}} \quad \forall x \in \mathbb{R}$$

Prove "(i) $f'(0)$ exists and

(ii) $y = 3x+2$ is a tangent line.

pf

(i) Q: $\displaystyle\lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$ exists ?

By $\circledast$ (evaluate $x=0$),

$$|f(0) - (\underbrace{3 \times 0 + 2}_{2})| \leq 0^{\frac{3}{2}} = 0$$

$$\Rightarrow \quad f(0) = 2$$

$$\frac{f(0+h) - f(0)}{h} = \frac{f(h) - 2}{h}$$

By $\circledast$,

By ⋯,

$$\left| f(h) - (3h+2) \right| \le |h|^{\frac{3}{2}} = |h| \cdot |h|^{\frac{1}{2}}$$

$$\| $$

$$\left| (f(h) - 2) - 3h \right|$$

$$\Rightarrow \quad \left| \frac{f(h) - 2 - 3h}{h} \right| \le \underbrace{|h|^{\frac{1}{2}}} \to 0$$

as $h \to 0$

$$\| $$

$$\left| \frac{f(h) - 2}{h} - 3 \right|$$

By Pinching Thm,

$$\lim_{h \to 0} \left| \frac{f(h) - 2}{h} - 3 \right| = 0$$

$$\Rightarrow \quad \lim_{h \to 0} \frac{f(h) - 2}{h} = 3$$

$$\| $$

$$f'(0)$$

so $f'(0)$ exists, $= 3$ — (i)

The tangent line of graph of f at $(0, f(0)) = (0, 2)$ is

$$(y-2) = f'(0) \cdot (x-0)$$

$$\overset{\shortparallel}{y-2} \qquad \overset{\shortparallel}{3 \cdot x}$$

i.e. $y = 3x + 2$ —(ii) #

Q7:

"$\infty - \infty$"

$$\lim_{x \to 0} \left(\underbrace{\frac{1}{x^2}}_{\to \infty} - \underbrace{\frac{1}{\sin^2 x}}_{\to \infty} \right)$$

$$= \lim_{x \to 0} \frac{\sin^2 x - x^2}{x^2 \sin^2 x} \qquad \begin{matrix} \to 0 - 0 = 0 \\ \to 0 \end{matrix}$$

"$\frac{0}{0}$"

$$= \lim \frac{(\sin^2 x - x^2)'}{(x^2 \sin^2 x)'}$$

$$\lim_{x\to 0} \quad (x^2 \sin^2 x)$$

$$= \lim_{x\to 0} \frac{2\sin x\cdot\cos x - 2x}{2x\sin^2 x + x^2\cdot 2\sin x\cdot\cos x}$$

$$= \lim_{x\to 0} \frac{2\cos x\cdot\cos x - 2\sin x\cdot\sin x - 2}{2\cdot\sin^2 x + 2x\cdot 2\sin x\cdot\cos x + 2x\cdot 2\sin x\cdot\cos x + x^2\cdot 2\cos^2 x - x^2\cdot 2\sin^2 x}$$

$$= \lim_{x\to 0} \frac{2\cos 2x - 2}{2\sin^2 x + 4x\sin 2x + 2x^2\cos 2x}$$

$= \dots$

<u>Another sol:</u>

$$\lim_{x\to 0} \frac{\sin^2 x - x^2}{x^2\sin^2 x} = \lim_{x\to 0} \frac{\sin^2 x - x^2}{x^4}\cdot\overbrace{\left(\frac{x}{\sin x}\right)^2}^{1}$$

$$\lim \frac{\sin^2 x - x^2}{x^4} = \lim_{x\to 0} \frac{\overbrace{2\sin x\cdot\cos x}^{\sin 2x} - 2x}{4x^3}$$

$x \to 0$

$$= \lim_{x \to 0} \frac{2\cos 2x - 2 \quad \to 0}{12x^2 \quad \to 0} = \lim_{x \to 0} \frac{-4\sin 2x \quad \to 0}{24x \quad \to 0}$$

$$= \lim_{x \to 0} \frac{-8\cos 2x}{24} = -\frac{1}{3}$$

So

$$\text{ans} = -\frac{1}{3} \cdot 1 = -\frac{1}{3} \ \#$$